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A stochastic *Feline* immunodeficiency virus model with vertical transmission

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HIGHLIGHTS

- A stochastic Feline immunodeficiency virus model (FIV) with vertical transmission is developed.
- The stochastic extinction and persistence of the FIV are given.
- The vertical transmission may be beneficial to the persistence of the FIV.
- Decreasing the vertical transmission is useful to control the spread of the FIV.

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ABSTRACT

In this paper, we formulate a stochastic Feline immunodeficiency virus (FIV) model with vertical transmission to investigate the effect of environmental fluctuations on the FIV dynamics. We prove that the threshold parameter $\mathcal{R}_0^s$ can be used to identify the stochastic extinction and persistence of the FIV: if $\mathcal{R}_0^s < 1$, the FIV will be extinct a.s., while if $\mathcal{R}_0^s > 1$, the FIV will persist a.s. Epidemiologically, we find that large environmental fluctuations can suppress the outbreak of FIV, and the vertical transmission may be beneficial to the persistence of the FIV. Thus, in order to control the spread of the FIV, we must decrease the vertical transmission.

1. Introduction

Feline immunodeficiency virus (FIV) causes an immune deficiency in cats that is very similar to the acquired immune deficiency syndrome in humans [1]. The lentivirus infections of cats (FIV) appear to bear the closest similarity in their pathogenesis to HIV infection and AIDS [2]. In order to understand the dynamics of the FIV, Ducrot et al. [3] established the following model:

$$\begin{cases} \dfrac{dS}{dt} = bS + \theta b_I I - (m + k(S+I))S - \beta SI, \\ \dfrac{dI}{dt} = \beta SI + (1-\theta) b_I I - (m + k(S+I))I - \alpha I, \\ S(0) = S_0 > 0, \ \ I(0) = I_0 > 0, \end{cases} \tag{1.1}$$

where $S(t)$ and $I(t)$ are susceptible and infectious cats, respectively, and $N(t) = S(t) + I(t)$. All parameters are nonnegative, b and b_I are the natural birth rate for $S(t)$ and $I(t)$, respectively, and $b_I(0 \le b_I \le b,)$ implies the vertical transmission, $0 \le \theta \le 1$ is the proportion of offspring born from an infective individual that is susceptible at birth, $1/\alpha > 0$ the average time spent in the infectious class, $m + kN$ the mortality rate of the cat population N, and m the natural death rate, $b - m > 0$

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the intrinsic growth rate of cat populations $N(t)$ in the absence of resource limits, βSI the bilinear horizontal transmission, and β the effective per-capita contact rate of infective individuals.

In [3], Ducrot et al. showed that model (1.1) admits a unique endemic equilibrium $E^* = (S^*, I^*)$ with $0 < S^*, I^* < \dfrac{b-m}{k}$, which is globally asymptotically stable if $\theta > 0$ and

$$T_0^{dd} := \frac{\beta(b-m)}{k[b+\alpha-(1-\theta)b_I]} > 1.$$

Simple calculations show that, $T_0^{dd} > 1$ is equivalent to the basic reproduction number

$$\mathcal{R}_0 := \frac{\beta(b-m)}{k(b+\alpha)} + \frac{(1-\theta)b_I}{b+\alpha} > 1. \tag{1.2}$$

On the other hand, stochastic noise plays an indispensable role in transmission of diseases, especially in a small total population. And it seems more practical to consider stochastic epidemic models [4–19]. For studying the effect of environmental variability on the virus dynamics of the FIV, based on the results of [3], Wang and co-workers [20,21] studied the following stochastic differential equations (SDE) model

$$\begin{cases} dS = [bS - (m + k(S+I))S - \beta SI]dt - \sigma SIdB(t), \\ dI = [\beta SI + b_I I - (m + k(S+I))I - \alpha I]dt + \sigma SIdB(t), \end{cases} \tag{1.3}$$

where $B(t)$ is the standard independent one-dimensional Wiener process defined over the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets).

Obviously, model (1.3) is a stochastic version corresponding to the special case of $\theta = 0$ of model (1.1). And there naturally comes a question: what is the effect of θ on the stochastic dynamics of the FIV? The main focus of this paper is to investigate the virus dynamics of the following SDE model corresponding to the deterministic model (1.3):

$$\begin{cases} dS = [bS + \theta b_I I - (m + k(S+I))S - \beta SI]dt - \sigma SIdB(t), \\ dI = [\beta SI + (1-\theta)b_I I - (m + k(S+I))I - \alpha I]dt + \sigma SIdB(t). \end{cases} \tag{1.4}$$

The rest of this paper is organized as follows: In Section 2, we show the existence and uniqueness of the global positive solution of (1.4). In Section 3, we provide the conditions which will cause disease to die out. In Section 4, by means of Lyapunov function, we prove that the persistence of the infection of model (1.4). In the last section, Section 5, we provide a brief discussion and summary of main results.

2. Existence and uniqueness of the global positive solution

Theorem 1. *For any initial condition $(S_0, I_0) \in \mathbb{R}_+^2$, there is a unique solution $(S(t), I(t))$ of the SDE model (1.4) for all $t \geq 0$ and the solution will remain in $\mathbb{R}_+^2$ with probability one, namely, $(S(t), I(t)) \in \mathbb{R}_+^2$ for all $t \geq 0$ almost surely (a.s.).*

Proof. Since the coefficients of the SDE model (1.4) are locally Lipschitz continuous, for any initial value $(S_0, I_0) \in \mathbb{R}_+^2$, there is a unique local solution $(S(t), I(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global in $\mathbb{R}_+^2$, we need to show that $\tau_e = \infty$ a.s. We choose a sufficiently large non-negative number r_0 such that both of S_0 and I_0 lie in the interval $[1/r_0, r_0]$. For each integer $r \geq r_0$, we can define the stopping time

$$\tau_r = \inf\{t \in [0, \tau_e) : S(t) \notin (1/r, r) \text{ or } I(t) \notin (1/r, r)\},$$

where $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_r is increasing as $r \to \infty$. Set $\tau_\infty = \lim_{t\to\infty} \tau_r$, then $\tau_\infty \leq \tau_e$ a.s.

To prove that $\tau_e = \infty$, it is sufficient to prove that $\tau_\infty = \infty$ a.s. If possible, let us assume that the statement is false. Then there exist two constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon. \tag{2.1}$$

Hence we can find an integer $r_1 \geq r_0$ such that

$$P\{\tau_r \leq T\} \geq \epsilon. \tag{2.2}$$

for all $r \geq r_1$. Define a C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$V(S, I) = (S + 1 - \log S) + (I + 1 - \log I).$$

The positivity of $V(\cdot)$ for all $(S, I) \in \mathbb{R}_+^2$ can be obtained by the fact $(z + 1 - \log z) \geq 0$ for all $z > 0$. Calculating the differential of V along the solution trajectories of the system (1.4) by using Itô's formula, we get

$$\begin{aligned} dV &\leq [bS - b + \theta b_I I + 2m + 2k(S+I) + \beta I + b_I I - b_I + \theta b_I + \alpha + \sigma^2]dt \\ &\quad -\sigma I(S-1)dB_1 + \sigma S(I-1)dB_2 \\ &\leq [(2m + \alpha + \sigma^2 + (2k+b))S + (\beta + 2k + (1+\theta)b_I)I]dt - \sigma I(S-1)dB + \sigma S(I-1)dB. \end{aligned} \tag{2.3}$$