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Bistability and Turing pattern induced by cross fraction diffusion in a predator–prey model[☆]

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HIGHLIGHTS

- The model exhibits bistability phenomenon.
- Cross fractional diffusion can create Turing patterns.
- The smaller order of fractional diffusion is, the more easily Turing instability occurs.

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ABSTRACT

In this work, we study a diffusive predator–prey model with mutual interference among the predators while searching for food. We prove that the model exhibits bistability, which indicates that there is no patterns for our model. When proper cross fractional diffusion terms are introduced in the model, the Turing pattern emerges when cross fractional diffusion coefficients fall into some domain.

1. Introduction

Volterra (1926) first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. Since then, qualitative and quantitative analysis on the predator–prey model is of practical and theoretical significance and has been an important area in ecology and mathematical biology (see, e.g. [1]).

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In [2], Wang et al. considered a predator–prey model in a bounded domain $\Omega \subset \mathbb{R}^N$ with no-flux boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = u(\alpha - \beta u) - \dfrac{cuv}{mv+1}, & (x,t) \in \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = \dfrac{suv}{mv+1} - rv, & (x,t) \in \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x) \geq 0, \ v(x,0) = v_0(x) \geq 0, \ x \in \Omega. \end{cases} \tag{1.1}$$

Here $u(x,t)$ and $v(x,t)$ stand for the density of prey and predator, respectively. $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and ν is the outward unit normal on $\partial\Omega$. The no-flux boundary condition indicates that there is zero population flux across the boundary. Model (1.1) is based on the following assumptions:

(A1) The growth rate of the prey in the absence of the predator is the traditional logistic form: $u(\alpha - \beta u)$.
(A2) The functional response of the predator is $\frac{cu}{mv+1}$. c is the capture rate [3] and m denotes a reduction in the predation rate at high predator densities due to mutual interference among the predators while searching for food (see [4,5]).
(A3) s is the conversion rate and r represents the natural death rate of the predator. All parameters of the reaction term are positive, the diffusion coefficient $d_i \geq 0$ $(i = 1, 2)$.

System (1.1) has the trivial steady state $(0, 0)$ and the semi-trivial steady state $\left(\frac{\alpha}{\beta}, 0\right)$. It has a positive constant steady state if and only if $s\alpha > r\beta$, in which case it is uniquely given by

$$u^* = \frac{s(m\alpha - c) + \sqrt{s^2(m\alpha - c)^2 + 4cmrs\beta}}{2ms\beta}, \qquad v^* = \frac{su^* - r}{mr}.$$

Wang et al. [2] proved that the positive steady state (u^*, v^*) is globally asymptotically stable for the corresponding ODE model of (1.1). Taking into account additive Allee effect of the prey, they further investigated dynamical behavior of (1.1) with additive Allee effect. The authors found that under some conditions, additive Allee effect and diffusion together can produce Turing instability.

In 1952, Turing [6] suggested that interacting chemicals at a homogeneous steady state can be destabilized by spatial diffusion, as explored via a system of two coupled reaction–diffusion equations. This kind of instability is called the Turing instability or the diffusion driven instability. Starting with the Turing's idea, spatial and temporal pattern formations of interacting species in biological, social, chemical, hydrodynamical system, etc., have been standing as a central object of research in recent decades (see e.g. [7–14]).

One of the purpose of this article is to further explore Turing's diffusion induced instability for the corresponding cross fractional diffusion system of (1.1). In Section 2, we will show that system (1.1) has a bistable phenomenon, that is, when $s\alpha \leq r\beta$, there is no positive constant steady state, and the semi-trivial steady state $\left(\frac{\alpha}{\beta}, 0\right)$ is globally asymptotically stable. Thus the predator population cannot evade extinction while the prey population stabilizes at the level $\frac{\alpha}{\beta}$. But if $s\alpha > r\beta$, a unique positive constant steady state denoted by (u^*, v^*) exists and is globally asymptotically stable . This suggests that Turing instability does not occur for system (1.1). In Section 3, we show that cross fractional diffusion makes the unique positive constant steady state unstable.

2. Global asymptotical stability

In this section we prove the global asymptotical stability of the semi-trivial steady state $\left(\frac{\alpha}{\beta}, 0\right)$ and the positive constant steady state (u^*, v^*).

We first show a lemma.

Lemma 2.1. *Let a, b be positive constants, $\phi \in C^1([a, \infty))$ and ϕ be bounded from below. Suppose that $\psi \geq 0$, $\int_a^\infty h(t)dt < \infty$ and*

$$\phi'(t) \leq -b\psi(t) + h(t). \tag{2.1}$$

If either $\psi \in C^1([a, \infty))$ and $\psi'(t) \leq k$ in $[a, \infty)$ for some constant $k > 0$, or $\psi \in C^\delta([a, \infty))$ and $\|\psi\|_{C^\delta([a,\infty))} \leq k$ for some constants $0 < \delta < 1$ and $k > 0$, then $\lim_{t\to\infty} \psi(t) = 0$.

Notice that system (1.1) has a unique non-negative global solution (u, v). By the maximum principle we know that if $u_0(x) \not\equiv 0$ and $v_0(x) \not\equiv 0$, then $u(x,t) > 0$ and $v(x,t) > 0$ on $\overline{\Omega}$ for $t > 0$. It is easy to show from the first equation of (1.1) that

$$0 < u(x,t) \leq \max\left\{\frac{\alpha}{\beta}, \max_{\overline{\Omega}} u_0(x)\right\} := A, \ \forall x \in \overline{\Omega}, \ t \geq 0. \tag{2.2}$$