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General solution of the Black–Scholes boundary-value problem

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HIGHLIGHTS

- We present infinitely many solutions of the Black–Scholes boundary problem.
- The Black–Scholes option valuation formula is included as a special solution.
- The solutions consist of many independent functions, involving Hermite polynomials.
- The Black–Scholes boundary-value problem violates the law of one price.

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ABSTRACT

The Black–Scholes formula for a European option price, which resulted in the 1997 Nobel Prize in Economic Sciences, is known to be the unique solution of the boundary-value problem consisting of the Black–Scholes partial differential equation and the terminal condition defined by the European call option. This has been one of the most popular tools of finance in theory as well as in practice. Here we present infinitely many solutions of the boundary value problem, involving Hermite polynomials. This indicates that the Black–Scholes boundary-value problem violates the law of one price, which is one of the fundamental concepts in economics.

1. Introduction

The Black–Scholes formula for a European option price is known to be a unique solution to the Black–Scholes partial differential equation with the terminal condition corresponding to the European option [1]. This resulted in the 1997 Nobel Prize in Economic Sciences, and has served as a paradigmatic tool of finance in theory as well as in practice [2].

To solve the Black–Scholes partial differential equation, one may conveniently consider the inverted time, in terms of which the Black–Scholes equation takes the form of the heat equation. Here we remark that the terminal condition is not differentiable and the terminal time is excluded. Related to this, we also point out that the inverted problem of the Black–Scholes equation is not exactly the same as the standard initial-value problem.

Making use of this, we show that there exist infinitely many solutions to the Black–Scholes partial differential equation with the terminal condition. Such solutions include the Black–Scholes option valuation formula as a special one. Among additional solutions, in particular, there also exist solutions displaying discontinuity, which reflects the singularity in the terminal condition. Existence of such many solutions implies that the Black–Scholes partial differential equation for the European option violates the well-known law of one price, which is one of the fundamental concepts in economics [3,4].

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2. Black–Scholes formula

Under the Black–Scholes environment, Black and Scholes [1] considered a European call option that pays $[x_T - K]^+ \equiv \max\{x_T - K, 0\}$ at expiration date T, where x_t is the stock price at time t and the striking price K is a positive constant. They showed that its fair value $w(x_t, t)$ satisfies the partial differential equation (PDE):

$$\partial_t w(x,t) = rw(x,t) - rx\partial_x w(x,t) - \frac{1}{2}v^2x^2\partial_x^2 w(x,t) \tag{1}$$

for $0 \le t < T$ and $x > 0$, where r is the risk-free interest rate, v is the volatility of the stock, and $\partial_t w(x,t) \equiv \partial w(x,t)/\partial t$, etc. with the subscript t on x suppressed for simplicity. This is nothing but Eq. (7) of Ref. [1] and is nowadays called the Black–Scholes PDE. By the definition of the European call option, its price satisfies the terminal condition (in the limit $t \to T$ from below):

$$\lim_{t\uparrow T} w(x,t) = [x-K]^+ \tag{2}$$

for $x \neq K$.

It is well known that the Black–Scholes PDE reduces, via an appropriate transformation, to the heat equation, as sketched below: Defining the inverted time $\tau \equiv T - t$ and letting $u \equiv \ln(x/K)$, we write the price function depending on u and τ in the form $w(x,t) \equiv K\tilde{w}(u,\tau)$. We further define $f(u,\tau) \equiv \tilde{w}(u,\tau)e^{-\alpha u - \beta\tau}$, where $\alpha \equiv -\frac{1}{v^2}\left(r - \frac{v^2}{2}\right)$ and $\beta \equiv -\frac{1}{2v^2}\left(r + \frac{v^2}{2}\right)^2$. It is then straightforward to show that Eq. (1) leads to

$$\frac{\partial f(u,\tau)}{\partial \tau} = \frac{v^2}{2}\frac{\partial^2 f(u,\tau)}{\partial u^2}, \tag{3}$$

which is nothing but the heat equation. Accordingly, the solution of the Black–Scholes PDE can be expressed in terms of the solution of the heat equation. In general, given the initial condition $f(u, \tau = 0) = g(u)$, the solution of Eq. (3) reads

$$f(u,\tau) = \int_0^u du' G(u,u';\tau)g(u'), \tag{4}$$

where $G(u, u'; \tau)$ is the propagator (or Green's function), i.e., a solution with the initial condition given by a delta function, $G(u, u'; \tau{=}0) = \delta(u - u')$. Specifically, the propagator takes the Gaussian form

$$G(u,u';\tau) = \frac{1}{\sqrt{2\pi v^2\tau}} e^{-\frac{(u-u')^2}{2v^2\tau}}, \tag{5}$$

which is the "fundamental solution" of the heat equation, and from the solution $f(u,\tau)$, one obtains the solution $w(x,t)$ of the Black–Scholes PDE, via $w(x,t) = K^{1-\alpha}x^\alpha f(\ln(x/K), T{-}t)e^{\beta(T-t)}$.

Black and Scholes then discussed the uniqueness of the solution of the boundary-value problem consisting of the Black–Scholes PDE (1) and the terminal condition (2) as follows:

> There is only one formula $w(x,t)$ that satisfies the differential equation (1) subject to the boundary condition (2). This formula must be the option valuation formula.

Specifically, the solution is given by

$$w^{BS}(x,t) = xN(d_1) - Ke^{-r\tau}N(d_2), \tag{6}$$

where $N(d) \equiv (2\pi)^{-1/2}\int_{-\infty}^{d} dz\, e^{-z^2/2}$ is the cumulative distribution function of the standard normal random variable and

$$d_1 \equiv \frac{1}{v\sqrt{\tau}}\left[\ln\frac{x}{K} + \left(r + \frac{v^2}{2}\right)\tau\right]$$
$$d_2 \equiv \frac{1}{v\sqrt{\tau}}\left[\ln\frac{x}{K} + \left(r - \frac{v^2}{2}\right)\tau\right]. \tag{7}$$

Eq. (6) is the Black–Scholes formula for the European call option price, which is also known as the Black–Scholes–Merton formula to acknowledge the valuable contributions by Merton [5]. For details, the readers are referred to Ref. [6].

3. Generalized solution

As commented in Section 2, the time-inversion of the Black–Scholes PDE is closely related to the heat equation. Accordingly, the solution of the Black–Scholes PDE is obtained from the heat equation, the general solution of which is given by Eq. (4). Note, however, that the inverted problem of the Black–Scholes PDE is not exactly the same as the standard initial-value problem: In particular, the time interval $0 \le t < T$, along with the constraint $x \neq K$ for the validity of the Black–Scholes PDE, transforms into the interval $0 < \tau \le T$. Namely, the initial time $\tau = 0$ is excluded and instead the