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A class of generalized Ginzburg–Landau equations with random switching

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ABSTRACT

This paper focuses on a class of generalized Ginzburg–Landau equations with random switching. In our formulation, the nonlinear term is allowed to have higher polynomial growth rate than the usual cubic polynomials. The random switching is modeled by a continuous-time Markov chain with a finite state space. First, an explicit solution is obtained. Then properties such as stochastic-ultimate boundedness and permanence of the solution processes are investigated. Finally, two-time-scale models are examined leading to a reduction of complexity.

1. Introduction

This paper focuses on a class of generalized Ginzburg–Landau equations, namely, stochastic Ginzburg–Landau equations in random environment modeled by a random switching process. In contrast to the well-known Ginzburg–Landau equations with random disturbances considered in the literature, higher order nonlinearity is allowed. More importantly, the systems are in a random environment that is modeled as a random discrete event process given by a switching process. Thus, the system under consideration may be considered as a hybrid system in which continuous dynamics and discrete events coexist. Our effort in this paper is devoted to obtaining existence and uniqueness of solutions, permanence of solutions, and reduction of complexity using two-time-scale formulation.

In 1950, Ginzburg and Landau proposed a class of deterministic differential equations to describe phase transitions for superconductivity in [1]. They observed the existence of two types of superconductors depending on the energy of the interface between the normal and superconducting states. Their paper has led to significant developments to the nowadays known Ginzburg–Landau theory. Because of its prevalence in applications, this class of equations has been attracting much attention in the past decades. For instance, Ginzburg–Landau equations have been used in many areas including the theory of bistable systems, chemical turbulence, phase transitions in non-equilibrium systems, nonlinear, optics with dissipation, thermodynamics, and hydrodynamic systems, etc.; see [2–4] and references there in.

Because random noise is often unavoidable, taking into consideration of stochastic disturbances is necessary. To account for the noise effect, stochastic Ginzburg–Landau equations have received much attention in recent years. For example, Neiman and Geier [5] studied stochastic resonance in an over-damped bistable system driven by white and harmonic noises. In [6,7], delay stochastic Ginzburg–Landau equations were considered, whose solutions describe the stochastic evolution of

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the position of a particle trapped in a double well potential in the presence of a time delayed force and Gaussian white noise. Subsequently, in [8], Kloeden and Platen provided an explicit solution to the Ginzburg–Landau equation given by

$$dx(t) = [(a + \frac{\sigma^2}{2})x(t) - bx^3(t)]dt + \sigma x(t)dW(t), \tag{1.1}$$

where $W(t)$ is a standard Brownian motion, a, σ, and $b > 0$ are constants. Dung [9] studied a number of qualitative properties of the solution to a stochastic Ginzburg–Landau equation with impulsive effects.

Because of the rapid growth in science and technology, networked systems come into being. The new challenges call for more sophisticated mathematical models. One of the important advances is the development of novel models of hybrid systems in which continuous dynamics and discrete events coexist and interact. To consider both random uncertainty due to Brownian motion type of disturbances and stochastic environment represented by jump processes taking values in a finite set, switching diffusions have gained much needed attention recently; see [10,11] among others. Such switching dynamic systems are nonlinear stochastic systems with another stochastic source depicting the random environment changes as switching processes. The presence of both continuous dynamics and discrete events enables one to describe complex systems and their inherent uncertainty and randomness in the environment effectively.

Aiming at enlarging the applicability and suitability for a wider range of problems, this paper focuses on a class of real-valued systems known as generalized stochastic Ginzburg–Landau equations with regime switching. It is a generalized model since the nonlinear terms have faster growth rates than that of the usual Ginzburg–Landau equations. More importantly, we use a randomly switching process to model stochastic environment and other random factors that are not covered in the usual stochastic differential equations.

The rest of the paper is organized as follows. Section 2 presents the generalized Ginzburg–Landau equations with switching that we wish to study. Also derived in this section is an explicit solution. Section 3 investigates properties such as stochastically ultimate boundedness and permanence of the solution processes. These results may shed some light for the subsequent study on superconductivity and other desired properties. Section 4 examines a class of systems with two-time scales. The main idea here is to reduce the computational complexity. Finally, Section 5 gives some further remarks to conclude the paper.

2. Formulation and existence of solution

2.1. Formulation

Let $W(\cdot)$ be a real-valued Brownian motion, and $\alpha(\cdot)$ be a continuous-time Markov chain that is independent of $W(\cdot)$ with a state space $\mathcal{M} = \{1, \dots, m\}$ and generator $Q = (q_{ij})$. Recall, Q satisfies the conditions $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{m} q_{ij} = 0$ for each $i \in \mathcal{M}$. Note that for the continuous-time Markov chain $\alpha(t)$,

$$P\{\alpha(t+\delta) = j | \alpha(t) = i\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } i = j. \end{cases}$$

The objective of this paper is to treat the generalized Ginzburg–Landau equations with random switching in which the coefficients of the systems depend on an additional time variable. Thus the coefficients of the systems are time varying in addition to the time-varying and jump properties due to the Markov chain. Consider the equation

$$dX(t) = \Big[a(t, \alpha(t))X(t) - b(t, \alpha(t))X^{k+1}(t)\Big]dt + \sigma(t, \alpha(t))X(t)dW(t), \tag{2.1}$$

where $k \geq 2$ is an integer. It then follows that the associated generator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}f(t, x, i) \quad &= \frac{\partial f(t, x, i)}{\partial t} + (a(t, i)x - b(t, i)x^{k+1})\frac{\partial f(t, x, i)}{\partial x} \\ &\quad + \frac{1}{2}\sigma^2(t, i)x^2 \frac{\partial^2 f(t, x, i)}{\partial x^2} + \sum_{j=1}^{m} q_{ij} f(t, x, j), \end{aligned} \tag{2.2}$$

for each $i \in \mathcal{M}$, where $f(\cdot, \cdot, \cdot) : [0, \infty) \times \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ such that for each $i \in \mathcal{M}$, $f(\cdot, \cdot, i) \in C^{1,2}$. That is, f has continuous partial derivative with respect to t, and continuous partial derivative with respect to x up to the second order.

2.2. Explicit solution

In this section, we demonstrate that (2.1) has a global explicit solution that is positive for $t \geq 0$.

(A1) For each $i \in \mathcal{M}$, $a(t, i)$, $b(t, i)$ and $\sigma(t, i)$ are bounded integrable functions defined on $[0, +\infty)$ and $b(t, i) \geq 0$.

Theorem 2.1. *Assume (A1). Then for any initial condition $x_0 := X(0) > 0$, there is a unique positive solution of (2.1) on $t \geq 0$ explicitly given by*

$$X(t) = \frac{\exp(\Gamma(t))}{\Big[\frac{1}{x_0^k} + k\int_0^t b(s, \alpha(s))\exp(k\Gamma(s))ds\Big]^{\frac{1}{k}}}, \tag{2.3}$$