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Entanglement and thermodynamics in non-equilibrium isolated quantum systems

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ABSTRACT

In these lectures, I pedagogically review some recent advances in the study of the non-equilibrium dynamics of isolated quantum systems. In particular I emphasise the role played by the reduced density matrix and by the entanglement entropy in the understanding of the stationary properties after a quantum quench. The idea that the stationary thermodynamic entropy is the entanglement accumulated during the non-equilibrium dynamics is introduced and used to provide quantitative predictions for the time evolution of the entanglement itself. The harmonic chain is studied as an elementary model in which the quench dynamics can be easily and exactly worked out. This example provides a useful playground where general concepts can be simply understood and later applied to more complex and realistic systems.

1. Introduction

During the last decade there has been an intensive theoretical and experimental activity aimed to understand the non-equilibrium dynamics of isolated many-body quantum systems. Among the important questions that these investigations have been trying to answer, we will focus in understanding whether is possible that for large time these systems can attain stationary properties, in which sense this is compatible with the unitary time evolution of quantum mechanics, and under what conditions these stationary properties are the same as in a statistical ensemble. For the interested reader, there are several excellent reviews [1–6] which consider almost all possible aspects of the non-equilibrium dynamics of isolated quantum systems. These reviews are much more complete of the present lecture notes which instead report only some introductory and elementary aspects of the problem since they are addressed to young students willing to have a first idea about the subject.

The problem we consider is that of a many-body quantum system prepared at time $t = 0$ in a non-equilibrium state $|\Psi_0\rangle$ and let evolve with a Hamiltonian H. At time t, the time evolved state is (we set $\hbar = 1$)

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle. \tag{1}$$

The initial state $|\Psi_0\rangle$ can be thought as the ground state of another Hamiltonian H_0 such that $[H, H_0] \neq 0$. We have in mind the situation in which one of the parameters of the Hamiltonian is suddenly changed at $t = 0$ and for this reason this non-equilibrium protocol is usually referred to as a quantum quench [7,8].

Uncountable theoretical and experimental investigations have shown that for large times and in the thermodynamic limit, the expectation values of a class of observables relax to stationary values (see e.g. the aforementioned reviews [1–6]). In some cases, these stationary values coincide with those in a thermal ensemble, although the dynamics governing the

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evolution is unitary and the initial state is pure. Indeed, the first study addressing the thermalisation of an isolated quantum system is by Von Neumann and dates back to 1929 [9]. One of the main reasons for the explosion of activity in the field during the last decade is the developing of experimental ultra-cold atom techniques presenting the unique feature of experimental control over interaction parameters, dimensionality, and isolation. These experiments provided pioneering results allowing to test the relaxation and the thermalisation of closed quantum systems [10–17].

These lecture notes are organised as follows. In Section 2 one cold-atomic experiment is briefly reviewed to show how relaxation and thermalisation indeed happen in isolated quantum systems. In Section 3 the concept of reduced density matrix is introduced to explain in which sense an isolated many-body quantum system can attain a stationary state. In Section 4 the entanglement entropy is defined and its role for the non-equilibrium dynamics is highlighted. In Section 5 the harmonic chain is considered as a simple example to study the quench dynamics and to test some of the ideas introduced earlier. Finally in 6 we explain the quasiparticle picture for the spreading of entanglement and show that, complemented with the knowledge of the stationary state, it could be used to obtain quantitative predictions for the time evolution of the entanglement entropy in a generic integrable model.

2. The quantum Newton cradle

The *quantum Newton cradle* [10] is the first cold atomic experiment showing the importance of the conservation laws for the non-equilibrium dynamics of isolated quantum systems. In this experiment, a cloud of a few hundred atoms of rubidium is initially prepared in a harmonic trapping potential in thermal equilibrium at a very low temperature (which can be assumed to zero for any practical purpose). The gas is dilute enough so that the interaction between two bosons can be considered point-like (i.e. a δ-function two-body interaction), but it is strong and not negligible. The cloud is split by a laser beam in two counter-propagating clouds with opposite momentum. The two clouds climb the harmonic potential in opposite direction up to when they reach the maximum value allowed by energy conservation; then they alight towards the centre of the trap where they interact and then climb again the potential; the process goes over and over for many times before relaxation takes place. The time evolution is essentially unitary during the probed time window.

The time evolution of the momentum distribution function is measured and averaged over several experimental realisations. The results show that this observable attains for large time a stationary distribution, but the stationary values depend on the dimensionality. In two and three dimensions, the system *relaxes very quickly and thermalises*; instead in one dimension, it *relaxes slowly to a non-thermal and unusual distribution*. The difference between the two cases relies on the fact that the one-dimensional system is (almost) integrable. Indeed it is nowadays widely accepted that generic systems thermalise and integrable ones relaxes to other statistical ensembles which take into account all the constraints imposed by the infinite number of conservation laws, as we shall see in the following section.

3. Stationary state and reduced density matrix

The experiment in the previous section is only one of the many examples showing that the stationary values of some observables after a quantum quench are described by statistical ensembles, although the initial state is pure and the time evolution is unitary. We need to clarify how we can accommodate this behaviour with the laws of quantum mechanics, i.e. we need to understand in which sense observables after a quantum quench can be described by a mixed state such as the thermal one.

The crucial concept to solve this apparent paradox is the reduced density matrix. This is defined as follows. Let us consider a non-equilibrium many-body quantum system (in arbitrary dimension) and bipartite it, i.e. consider two complementary *spatial* parts denoted as A and B respectively. Because of the unitarity of the time evolution, the entire system $A \cup B$ will always be in the pure state $|\Psi(t)\rangle$ in Eq. (1), but this is not the case if we focus our attention (and our measurements) on the subsystem A. The physics of the subsystem A is fully encoded in the reduced density matrix obtained by tracing over the degrees of freedom in B

$$\rho_A(t) = \mathrm{Tr}_B\big[\rho(t)\big], \tag{2}$$

where $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ denotes the density matrix of the entire system. Although $\rho(t)$ is a projector, $\rho_A(t)$ generically corresponds to a mixed state with non-zero entropy.

The reduced density matrix $\rho_A(t)$ is all we need to describe the correlation functions local within A: If we are interested in the expectation value of a product of local operators $\prod_i O(x_i)$ with $x_i \in A$ this is given by

$$\langle\Psi(t)|\prod_i O(x_i)|\Psi(t)\rangle = \mathrm{Tr}[\rho_A(t) O(x_i)]\,. \tag{3}$$

Thus as long as we are interested in local observables we do not have to retain information about the entire system, but we can limit to $\rho_A(t)$.

We are ready to understand in which sense a closed quantum system can relax to a stationary state [18–22]. If for *all* finite subsystems A embedded in an *infinite system*, the limit of the reduced density matrix for infinite time exists, i.e. if it

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