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Algorithm for multiplex network generation with shared links

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HIGHLIGHTS

- We introduce an algorithm to generate a multiplex networks with shared links.
- We theoretically provide an upper bound of the average degree of the shared network.
- We give the numerical verification of the algorithm on BA and random networks.
- Our proposed algorithm could generate multiplex networks in a rather general extent.

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ABSTRACT

Multiplex networks have been used to describe multilevel system by the way of combining several layers of sub-networks with one layer representing one sub-level system. Many multiplexes are characterized by a significant shared links in different layers. For this new network framework, research efforts have been paid on the study of its topological and dynamical properties. However, in these studies the network structures are mostly simple or specific, and the shared links in different layers has been mostly neglected, despite the fact that it is an ubiquitous phenomenon in most multiplexes. To systemically study multiplex network, a general multiplex network framework is necessary. In this work, we introduce an algorithm to generate a multiplex networks with shared links whose degree correlation functions of all its layers, the nodal degree correlation function between the layers, and the size of the resulting network are all tunable. Moreover, we give an upper bound of the average degree of the resulting network when the degree correlation functions introduced above are given and make efforts to maximize the average nodal degree of the multiplex networks. This algorithm may serve as a good candidate of standard technique for multiplex network generation.

1. Introduction

In the past years we have gained a large attention of the interplay between the topology of single complex networks [1] and the behavior of critical phenomena occurring on them [2,3]. Nevertheless, many systems are not formed by isolated networks; instead they are formed by a network of networks [4–6], which are composed of several sub-systems or are

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integrated at different levels. Complex networks, as a useful tool to describe complex systems have also been developed to describe such multilevel complex systems. We call them multiplex network, where individual nodes take part in several layers of networks simultaneously. From transportation networks to complex infrastructures, and to social and communication networks, a large variety of systems can be described in terms of multiplexes formed by a set of nodes interacting through different layers of networks. For example, in social networks where each individual node has different kinds of social ties; transportation systems where each location is connected to another location by different types of transport such as train and flight; the communication system includes traditional and modern medium; and the online communication system includes different sites such as Facebook and Twitter. Thus, nodes in the multiplex network could be connected with different types of links, and different types of links represent the connections in different sub-systems or in different level.

Very recently, several works have studied dynamical processes taking place on multiplexes and interacting networks and new surprising phenomena have been observed in this context, which include the study of robustness of cascading failure on interdependent network which is a concept intimate to multiplex network [7,8], the phase transition between the initial and the final fraction of failed nodes of the interdependent network [9,10], the diffusion dynamics on multiplex networks where non-trivial diffusion dynamics if observed which is not exist in the simplex network [11], the promotion effect of cascading dynamics in the multiplex network [12], the transportation strategy on the multi-layered network which is also a concept intimate to multiplex network [13], and a model proposed to generate a specific multiplex network [14,15]. Other empirical examples about the study of multiplex networks include multimodal transportation networks [16,17], climatic systems [18], economic markets [19], energy-supply networks [7] and the human brain [20]. Therefore, the offshoot of the network theory fundamental insights is that for us working in statistical mechanics it is now possible — in a sense it is mandatory to move into the field to shed light on the complexity on interdependent networks and multiplexes. In this context, new measures for multiplex [5,21–27] and new models of growing multiplexes [28–31] have been proposed.

However, the structures of multiplex networks studies so far are mostly simple or specific. To systemically study multiplex networks, a general framework of multiplex networks' generation is necessary so that we could study its topological and dynamical properties in a comprehensive scope. In this work, we will provide an algorithm by which we could generate a general multiplex network whose degree correlation functions of all its layers, the nodal degree correlation function between the layers, and the size of the network are all tuneable. In Section 2, we will introduce the multiplex network with shared links in detail and some definitions of the network structure to be used in the work. In Section 3, we will provide a condition which constrains what kind of multiplex networks can be generated and give the algorithm, and in Section 4 we will provide a modified algorithm by which we could generate a general multiplex network under constrained conditions. In addition, we maximize the average degree of the multiplex network and give the further discussion. We give the numerical verification of the algorithm in Section 5. And we conclude the paper in Section 6.

2. Model

A multiplex network could be generated by combining several simplex networks. In this paper, we consider two-layered networks, which are assigned as network *I* and network *II*. A common way of generating a multiplex network is as follows: first let the two simplex networks have the same size, denoted as N. If their sizes are different initially, we add isolated nodes to the network until they have the same network size. Then, we randomly choose two nodes which are in different simplex networks and merge them as one node. For this merged node, it has two types of degree. One is the degree of the parental node in network *I*, called type *I* degree, and the other is the degree of the parental node in network *II*, called type *II* degree. We repeat this procedure until all the N pairs of nodes are merged, and then the generated network who includes all the merged nodes and links is called the multiplex network.

For the generated multiplex network we could use a joint degree distribution $p^I(i, i')$ to denote the fraction of nodes who has i type *I* degree and i' type *II* degree. Besides, we may easily get the expression of the conditional probability $p^I(i'|i)$ which denotes the probability that a node with i type *I* degree has i' type *II* degree as $p^I(i'|i) = p^I(i, i')/p^D_{(1)}$. Moreover, the network *I* and the network *II* could be described by the degree distribution $p^D_{(1)}(k)$ and $p^D_{(2)}(k)$, the edge degree distribution $q_{(1)}(k)$ and $q_{(2)}(k)$, and the degree–degree distribution $p_{(1)}(i, k)$ and $p_{(2)}(i, k)$, respectively. The relations among them are

$$\sum_i p_{(1)}(i,k) = q_{(1)}(k) = kp^D_{(1)}(k)/\langle k\rangle_{(1)},$$
$$\sum_i p_{(2)}(i,k) = q_{(2)}(k) = kp^D_{(2)}(k)/\langle k\rangle_{(2)}, \tag{1}$$

where $\langle k\rangle_{(1)}$ and $\langle k\rangle_{(2)}$ are the average degree of network *I* and *II*, respectively, and $\langle k\rangle_{(1)}$ may not equal to $\langle k\rangle_{(2)}$. Moreover, in these distributions, $p^D_{(1)}(0)$ and $p^D_{(2)}(0)$ may exist and have non-zero values. However, $q_{(1)}(k)$ and $q_{(2)}(k)$ have no definition when $k = 0$ since the degree of a node at the end of a link could not be zero. Similarly, $p_{(1)}(i, k)$ and $p_{(2)}(i, k)$ have no definition when either i or k is zero. Thus, when $\langle k\rangle_{(1)}$, $\langle k\rangle_{(2)}$, $p_{(1)}(i, k)$ and $p_{(2)}(i, k)$ are given, with the normalization condition that $\sum_{k\geq 0} p^D_{(1)}(k) = \sum_{k\geq 0} p^D_{(2)}(k) = \sum_{k\geq 1} q_{(1)}(k) = \sum_{k\geq 1} q_{(2)}(k) = 1$, $p^D_{(1)}(k)$, $p^D_{(2)}(k)$, $q_{(1)}(k)$ and $q_{(2)}(k)$ can be deduced accordingly.

A new feature of the multiplex network is that some nodes could share both types of links. This situation happens when the two parental pair of nodes who merge to form two nodes in the multiplex network are connected to each other in their