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# SOLUTIONS OF FRACTIONAL LOGISTIC EQUATIONS BY EULER'S NUMBERS

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Abstract. In this paper, we solve in the convergence set, the fractional logistic equation making use of Euler's numbers. To our knowledge, the answer is still an open question. The key point is that the coefficients can be connected with Euler's numbers, and then they can be explicitly given. The constrained of our approach is that the formula is not valid outside the convergence set, The idea of the proof consists to explore some analogies with logistic function and Euler's numbers, and then to generalize them in the fractional case.

**Keywords:** Euler's numbers, Biological Application, Fractional logistic equation.
**AMS subject classification:** 11B68, 78A70, 26A33

## 1. Introduction

1.1. **The logistic function.** A logistic function is

$$u(t) = \frac{u_0}{u_0 + (1-u_0)e^{-t/M}} \tag{1.1}$$

with $M$ positive and $u_0$ positive and less than 1. The function was introduced by Pierre Francois Verhulst [28] to model the population growth. At the beginning of the process the growth of the population is fast; then, as saturation process begins, the growth slows, and then growth is close to be flat.

The logistic function is solution of the logistic differential equation

$$u'(t) = \frac{1}{M}(u(t) - u^2(t))$$

with initial condition

$$u(0) = u_0.$$

The key assumptions in the logistic model are:

- The population is composed by individual not distinguishable;
- The population is isolated;
- Self-limiting growth, that is an intrinsic mechanism of saturation holds when the density of population reaches a certain level.

These basic assumptions may be checked in laboratory for biological diffusion. At least for bounded time they agree to the experience, hence they may be adopted to describe phenomena as biological models of tumour growth ([1; 5; 8; 19]). As well many processes may be modelled by the logistic differential equation, or generalization of it, and the applications are wide and in different field of applications [24].

Sharing the applications with the logistic case, the fractional equation is a model for the growth of a given population, describing the population behaviour and showing an increase, a saturation and a flat asymptotic behaviour. The global shape is also respected by the fractional logistic case by numerical evidence; however, some peculiar differences show that the fractional model is a good candidate to model a memory effect on the population (see (1.7)) and that the fractional order may be modified along the process in order to constrain the growth (see for example [11] for the estimation of fractional order by observations).

The problem to give a solution of the fractional logistic equation was unsolved and several attempts have been done (see for instance [3; 15; 26; 29]). A close answer from an empirical point of view is contained in [22]. The solution we propose here has an exact representation given by a closed formula for the coefficients in the convergence set. The convergence analysis is stated in Theorem 3.1

A model considering a modified logistic equation has been also treated in [12].