Physica A 509 (2018) 1174–1182

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Performance of leader-following consensus on multiplex networks

Zhongxue Gan [a], Haibin Shao [b], Yuli Xu [a], Dewei Li [b,*]

[a] *ENN Science & Technology Development Co., Ltd, Langfang, Hebei, 065001, China*
[b] *Key Laboratory of System Control and Information Processing, Ministry of Education, Department of Automation, Shanghai Jiao Tong University, Shanghai, 200240, China*

HIGHLIGHTS

- Examine the correlation between the proportion of leader nodes and convergence rate of leader-following consensus on multiplex networks.
- The proportion of leaders plays a central role in the convergence rate of consensus for large-scale multiplex networks, regardless of the topological location of leaders and the topology of intralayer networks.
- The convergence rate of influenced consensus is monotonically increasing with respect to the proportion of leaders.
- Multiplex networks consisted of homogeneous networks are easier to control compared with that consisted of heterogeneous networks.
- The optimal control performance can be obtained by evenly selecting leader agents from each layer.
- Our findings shed light on the understanding of efficient propagation of both friendly and malicious influence exerted on multiplex networks.

ARTICLE INFO

Article history:
Received 11 October 2017
Received in revised form 9 May 2018
Available online xxxx

Keywords:
Multiplex network
Leader-following consensus
Convergence rate
Polynomial regression analysis

ABSTRACT

A multiplex network is derived from the interplay between the intralayer and interlayer networks. This paper examines the performance of controlling consensus dynamics on multiplex networks via leader-following approach. Regardless of the topological location of leaders and the topology of intralayer networks, it is shown that the proportion of leaders plays a central role in the convergence rate of leader-following consensus on multiplex networks, especially for large networks. We show that the convergence rate of leader-following multiplex consensus is monotonically increasing with respect to the proportion of leaders and its upper bound is invariant to the topology of the intralayer network. It turns out that the Erdős-Rényi random multiplex networks are easier to control compared with scale-free multiplex networks in terms of convergence rate of leader-following consensus. The polynomial regression analysis is also employed to fit the correlation between the proportion of leaders and the convergence rate of leader-following consensus on multiplex networks. Our findings shed light on the understanding of propagation of either friendly or malicious influence exerted on a multiplex network.

1. Introduction

The topology of many complex networks exhibits multiple layer structure, such as online social networks, transportation systems and protein–protein interactions etc. Kivelä et al. [1]. Multiplex networks are representations of multi-layer

* Corresponding author.
E-mail address: dwli@sjtu.edu.cn (D. Li).

https://doi.org/10.1016/j.physa.2018.06.049

interconnected complex networks where the nodes are the same at every layer. For instance, a social individual can have accounts of multiple online social networks such as Facebook, Twitter and Google plus; a person may also play multiple roles (or characters) in different communities (such as family, company, and club). Multi-modal urban transportation networks can also be mathematically abstracted as transportation dynamics on top of a multiplex structure. It has been shown that multiplexity has laid a broad impact on the structure and functionality of complex networks Kivelä et al. [1], Boccaletti et al. [2]. For instance, understanding brain connectivity from the interplay of functional relationships of brain activities and anatomical connections between brain areas is one of the most important problems in neuroscience Simas et al. [3]. Many complex systems can be modeled as interdependent network and one of its fundamental features is that the failure of the node in one network may trigger the failure of dependent nodes in neighboring networks Buldyrev et al. [4].

The functionality of a complex system derives from the interplay of local behavior of its components and the topology of inter-component interactions. The local behavior is often termed as protocol when each all the component behave in a similar manner. Consensus protocol on complex networks plays a paramount role in network diffusion, synchronization, and many distributed algorithms and has been extensively investigated Olfati-Saber and Murray [5], Ren et al. [6], Mesbahi and Egerstedt [7], Gomez et al. [8], Chen and Zhang [9], Zhang et al. [10]. It is well-known that the convergence rate of an autonomous consensus network can be quantified by the algebraic connectivity of a graph, i.e., the second smallest eigenvalue of graph Laplacian Mesbahi and Egerstedt [7]. However, the efficiency of controlling a consensus dynamics over a multiplex network can be characterized by the smallest eigenvalue of perturbed Laplacian matrix, where those nodes who are directly controlled are referred to as leaders. In this paper, we shall explore the effect of deployment of leaders in a multiplex consensus network on the convergence rate of consensus.

The organization of the paper is as follows. We first provide preliminaries and dynamical model in Section 2 and Section 3, respectively. The correlation between proportion of leaders and convergence rate of consensus is analyzed in Section 4. The concluding remarks are provided in Section 5.

2. Preliminaries

Denote by $\mathbb{R}$ and $\mathbb{N}$ as the real and natural numbers, respectively. A graph is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ with the node set $\mathcal{V} = \{1, 2, \ldots, n\}$, edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ satisfying $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The neighbor set of node i is $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. A multi-agent system is composed of multiple agents whose state at time $t \in \mathbb{R}_{\geq 0}$ is referred to as $x_i(t)$ for all $i \in \mathcal{V}$. The degree matrix $D \in \mathbb{R}^{n\times n}$ is such that $D = \textbf{diag}\{d_1, d_2, \ldots, d_n\}$, where $d_i = \sum_{j=1}^{n} a_{ij}$ represents the degree of node $i \in \mathcal{V}$. The Laplacian of the graph $\mathcal{G}$ is denoted by $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ whose elements are determined by $l_{ij} = \sum_{j=1}^{n} a_{ij}$ for $i = j$ and $l_{ij} = -a_{ij}$ for $i \neq j$. A subgraph $\mathcal{G}'$ is an induced subgraph if two nodes of $\mathcal{V}(\mathcal{G}')$ are adjacent in $\mathcal{G}'$ if and only if they are adjacent in $\mathcal{G}$. An $s \in \mathbb{N}$ length path $\mathcal{P}_s$ in graph $\mathcal{G}$ is an induced subgraph of graph $\mathcal{G}$ with node set $\mathcal{V}(\mathcal{P}) = \left\{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\right\}$ and edge set $\mathcal{E}(\mathcal{P}) = \left\{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots, (v_{i_{s-1}}, v_{i_s})\right\}$. The graph $\mathcal{G}$ is said to be connected if there exists at least one path between any two nodes in $\mathcal{V}(\mathcal{G})$. The direct sum of $s \in \mathbb{N}$ matrices $M_1, M_2, \ldots, M_s$ is denoted by

$$\bigoplus_{k=1}^{s} M_k = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_s \end{bmatrix}.$$

3. Leader-following multiplex consensus networks

Consider a multiplex network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ with $m \in \mathbb{N}$ layers and $n \in \mathbb{N}$ nodes in each layer. The ith node in kth layer is referred to as $i^{(k)}$ for all $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$. The intralayer topology is described by graphs $\mathcal{G}^{(k)} = \left(\mathcal{V}^{(k)}, \mathcal{E}^{(k)}, A^{(k)}\right)$ with adjacency matrix $A^{(k)} = [a_{ij}^{(k)}] \in \mathbb{R}^{n\times n}$ and Laplacian $\mathcal{L}^{(k)} = [l_{ij}^{(k)}] \in \mathbb{R}^{n\times n}$ where $k \in \{1, 2, \ldots, m\}$. The interlayer topology is characterized by graph $\mathcal{G}'$ with adjacency matrix $A' = [a'_{kl}] \in \mathbb{R}^{m\times m}$ and Laplacian $\mathcal{L}' = [l'_{ij}] \in \mathbb{R}^{m\times m}$. Nodes $i^{(k)}$ and $i^{(l)}$ are connected if and only if $a'_{kl} = 1$ for all $i \in \{1, 2, \ldots, n\}$, $k, l \in \{1, 2, \ldots, m\}$ and $k \neq l$. We assume that networks $\mathcal{G}^{(k)}$ are undirected, connected and without self-loop for all $k \in \{1, 2, \ldots, m\}$. For instance, a two-layered multiplex network is shown in Fig. 1. There are 5 nodes in each layer where interaction structures are characterized by $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$, respectively.

Denote the state of ith node in kth layer as $x_i^{(k)} \in \mathbb{R}$ for all $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$. Denote by $\boldsymbol{x}^{(k)} = [x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}]^\top \in \mathbb{R}^n$ as the state of kth layer and by

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{(1)} \\ \boldsymbol{x}^{(2)} \\ \vdots \\ \boldsymbol{x}^{(m)} \end{bmatrix} \in \mathbb{R}^{mn}$$