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HIERARCHIES OF N-POINT FUNCTIONS FOR NONLINEAR CONSERVATION LAWS WITH RANDOM INITIAL DATA

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Abstract. Nonlinear conservation laws subject to random initial conditions pose fundamental problems in the evolution and interactions of shocks and rarefactions. Using a discrete set of values for the solution, we derive a hierarchy of equations in terms of the states in two different methods. This hierarchy involves the n-point function, the probability that the solution takes on various values at different positions, in terms of the $(n+1)$-point function. In the first approach, these equations can be closed but the resulting solutions do not persist through shock interactions. In our second approach, the n-point function is expressed in terms of the $(n+1)$-point functions, and remains valid through collisions of shocks.

1. Introduction

1.1. **Background on Conservation Laws.** In the effort to create a mathematical description of turbulence, an important building block is the study of shocks and rarefactions together with random initial conditions. Although the model is a somewhat coarse description of turbulence in practice, Burgers' equation is extensively studied as a test case for new methods and types of randomness. It also possesses the surprising feature of producing discontinuous solutions, even from smooth initial data. From there it is then reasonable to seek broader classes of equations to which these properties can be extended.

Conservation laws arise in many physical problems. One interesting and protoypical case involves one-dimensional hard-point systems. In [17] predictions from the theory are outlined and confirmed using time-correlation functions, where one tracks the momentum and position of individual particles. Another application involves the Tracy-Widom distribution for growth processes [18]. Under certain conditions a rarefaction wave develops, while far away from this rarefaction, the initial equilibrium states are maintained. One can study these under invariant measures as in [23]. In hydrodynamics [26], one can study the Fokker-Planck equation for the one-particle distribution function and construct an equation of state indepedently, as an alternative to Monte-Carlo methods for gaseous flows. Conservation laws can also be applied to quantum field theory and studied to obtain a correct description of the stationary state [7] or to stochastic thermodynamics where probability fluxes in the system can be considered [22].

Thus, the statistics of shocks pose challenging problems in statistical mechanics and other fields. Moreover, there are many possible approaches to this set of problems. Just as the Ising and more general lattice models provided key prototypes for ferromagnets and equilibrium phase transitions, one can write basic conservation laws with random initial conditions that serve as fundamental building blocks for the study of shock statistics. Among these, as discussed below, are Burgers' equation and polygonal flux functions.

We develop kinetic equations for scalar conservation laws with a polygonal flux function. As mentioned above, the classical prototype for nonlinear conservation laws is Burgers' equation, which has been studied in numerous works (e.g. [1, 2, 4, 5, 3, 8, 10, 11]). This is a simple model that produces shocks, given by:

$$u_t + uu_x = 0,$$
$$u(x,0) = g(x). \tag{1.1}$$

This has also been studied extensively in Menon and Srinivasan [21] involving random initial conditions, which is directly relevant to our current work. Under conditions of sufficient regularity, the solution of a

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