Accepted Manuscript

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PII: S0378-4371(18)30318-2
DOI: https://doi.org/10.1016/j.physa.2018.03.010
Reference: PHYSA 19344

To appear in: *Physica A*

Received date : 18 December 2017
Revised date : 17 February 2018

Please cite this article as: P. Straka, Variable order fractional Fokker–Planck equations derived from Continuous Time Random Walks, *Physica A* (2018), https://doi.org/10.1016/j.physa.2018.03.010

Variable Order Fractional Fokker-Planck Equations derived from Continuous Time Random Walks

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Abstract

Continuous Time Random Walk models (CTRW) of anomalous diffusion are studied, where the anomalous exponent $\beta(x) \in (0, 1)$ varies in space. This type of situation occurs e.g. in biophysics, where the density of the intracellular matrix varies throughout a cell. Scaling limits of CTRWs are known to have probability distributions which solve fractional Fokker-Planck type equations (FFPE). This correspondence between stochastic processes and FFPE solutions has many useful extensions e.g. to nonlinear particle interactions and reactions, but has not yet been sufficiently developed for FFPEs of the"variable order" type with non-constant $\beta(x)$.

In this article, variable order FFPEs (VOFFPE) are derived from scaling limits of CTRWs. The key mathematical tool is the 1-1 correspondence of a CTRW scaling limit to a bivariate Langevin process, which tracks the cumulative sum of jumps in one component and the cumulative sum of waiting times in the other. The spatially varying anomalous exponent is modelled by spatially varying $\beta(x)$-stable Lévy noise in the waiting time component. The VOFFPE displays a spatially heterogeneous temporal scaling behaviour, with generalized diffusivity and drift coefficients whose units are length^2/time$^{\beta(x)}$ resp. length/time$^{\beta(x)}$. A global change of the time scale results in a spatially varying change in diffusivity and drift.

A consequence of the mathematical derivation of a VOFFPE from CTRW limits in this article is that a solution of a VOFFPE can be approximated via Monte Carlo simulations. Based on such simulations, we are able to confirm that the VOFFPE is consistent under a change of the global time scale.

Keywords: Anomalous Diffusion, Continuous Time Random Walk, Fractional Derivative, Variable Order, stochastic process limit, Lévy process
2010 MSC: 60F17, 60G22

1. Introduction

Subdiffusive processes are characterized by a sublinearly growing mean squared displacement proportional to t^β, $0 < \beta < 1$, and have been reported in many experimental systems [1, 2, 3, 4, 5]. For a majority of these systems, long rests of a walker are thought to be the main mechanism causing subdiffusion, and Continuous Time Random Walks (CTRWs) with heavy-tailed waiting times effectively capture this phenomenon [6, 7, 1]. CTRWs have become a widely used model, particularly because scaling limits of their probability distributions can be modelled by fractional differential equations and Fokker-Planck equations [8, 9, 10, 11] which can be extended tractably to model particle reactions [12, 13] and nonlinear interaction effects [14].

In the majority of the literature on CTRWs and FFPEs (fractional Fokker-Planck equations), the fractional parameter $0 < \beta < 1$ is treated as a global constant. The situation where β varies in space, however, is of interest in physics, since the strength of a trapping effect may vary throughout

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