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Dimensionally regularized Boltzmann–Gibbs statistical mechanics and two-body Newton's gravitation

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HIGHLIGHTS

- We study the q partition function of the Newtonian gravitation.
- We study the poles of this partition function in phase space.
- We use Dimensional Regularization of Bollini and Giambiagi.
- We use an analytic extension of Gradshteyn and Rizhik.
- We find interesting gravitational effects.

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ABSTRACT

It is believed that the canonical gravitational partition function Z associated to the classical Boltzmann–Gibbs (BG) distribution $\frac{e^{-\beta H}}{\mathcal{Z}}$ cannot be constructed because the integral needed for building up Z includes an exponential and thus diverges at the origin. We show here that, by recourse to (1) the analytical extension treatment obtained for the first time ever, by Gradshteyn and Rizhik, via an appropriate formula for such case and (2) the dimensional regularization approach of Bollini and Giambiagi's (DR), one can indeed obtain finite gravitational results employing the BG distribution. The BG treatment is considerably more involved than its Tsallis counterpart. The latter needs only dimensional regularization, the former requires, in addition, analytical extension.

1. Introduction

DR [1,2] constitutes one of the greatest advances in the theoretical physics of the last 45 years, with applications in several branches of physics (see, for instance, [3–56]).

It is commonly believed that the classical Boltzmann–Gibbs (BG) probability distribution cannot yield finite results because the associated partition function $\mathcal{Z}$ in ν dimensions diverges [57,58], as one has (m and M are the masses involved, G the gravitation constant, β the inverse temperature, and x-p the phase-space coordinates)

$$\mathcal{Z}_\nu = \int_M e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)} d^\nu x d^\nu p, \tag{1.1}$$

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with a positive exponential. However, such belief does not take into account the possibility of analytical extensions, that would take care of divergences, e.g., at the origin.

It has been shown in Ref. [59], for first time ever, that $\mathcal{Z}$ can be calculated for Tsallis entropy using the 40-years old DR technique.

Why are we insisting on this issue if it has been already solved? The issue needs revisiting because it does not work for $q = 1$, that is, for the Boltzmann–Gibbs statistics, due to the fact that we there face an exponential divergence. In this paper we report on how to overcome this problem by judicious use of an appropriate combination of DR plus analytical extension. This produces the first ever BG partition function for the two-body gravitational problem. We remark that the N-body gravitational problem has not yet been solved and constitutes a frontier research problem in Celestial Mechanics.

It is well known that, at a quantum field theory level, DR cannot cope with the gravitational field, since it is non-renormalizable. Our present challenge is quite different, though, because we deal with Newton's gravity at a *classical* level.

2. Analytic extension

In this section we collect a set of mathematical results that will be needed afterwards. This Section may be omitted at a first reading. We must now keep in mind that we are dealing with the integral of an exponentially increasing function given by (1.1). We resort to Ref. [60], and following it we consider a useful integral, that will greatly help with our inquires, after judicious specializations of it. This integral reads

$$\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{-\frac{\beta}{x}}dx = \beta^{\frac{\nu-1}{2}}\gamma^{\frac{\nu-1}{2}+\mu}\Gamma(1-\mu-\nu)e^{\frac{\beta}{2\gamma}}W_{\frac{\nu-1}{2}+\mu,-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right), \tag{2.1}$$

$|\arg(\gamma)| < \pi, \Re(1-\mu-\nu) > 0$, where W is *one* of the two Whittaker functions. One does not require $\Re\beta > 0$, as emphasized by Gradshteyn and Rizhik [60] (see figure in page 340, Eq. (7)), called ET II 234(13)a, where reference is made to [61] (Caltech's Bateman Project). The last letter "a" indicates that analytical extension has been performed. Choosing $\mu = 1$ above we find

$$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}}dx = \beta^{\frac{\nu-1}{2}}\gamma^{\frac{\nu+1}{2}}\Gamma(-\nu)e^{\frac{\beta}{2\gamma}}W_{\frac{\nu+1}{2},-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right), \tag{2.2}$$

valid for $\nu \neq 0, -1, -2, -3, \ldots$.. Additionally [60],

$$W_{\frac{\nu+1}{2},-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right) = M_{\frac{\nu+1}{2},\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right) = \left(\frac{\beta}{\gamma}\right)^{\frac{\nu+1}{2}}e^{-\frac{\beta}{2\gamma}}, \tag{2.3}$$

where M stands for the *other* Whittaker function. Thus,

$$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}}dx = \beta^{\nu}\Gamma(-\nu) \tag{2.4}$$

an integral that can be evaluated for all $\nu = 1, 2, 3, \ldots$. by recourse to the dimensional regularization technique [1,2]. Changing now β by $-\beta$ in (2.1) we have

$$\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{\frac{\beta}{x}}dx = (-\beta)^{\frac{\nu-1}{2}}\gamma^{\frac{\nu-1}{2}+\mu}\Gamma(1-\mu-\nu)e^{-\frac{\beta}{2\gamma}}W_{\frac{\nu-1}{2}+\mu,-\frac{\nu}{2}}\left(-\frac{\beta}{\gamma}\right). \tag{2.5}$$

Once again we choose $\mu = 1$ and have

$$\int_0^\infty x^{\nu-1}e^{\frac{\beta}{x}}dx = (-\beta)^{\frac{\nu-1}{2}}\gamma^{\frac{\nu+1}{2}}\Gamma(-\nu)e^{-\frac{\beta}{2\gamma}}W_{\frac{\nu+1}{2},-\frac{\nu}{2}}\left(-\frac{\beta}{\gamma}\right), \tag{2.6}$$

valid for $\nu \neq 0, -1, -2, -3, \ldots$.. One now faces

$$W_{\frac{\nu+1}{2},-\frac{\nu}{2}}\left(-\frac{\beta}{\gamma}\right) = M_{\frac{\nu+1}{2},\frac{\nu}{2}}\left(-\frac{\beta}{\gamma}\right) = \left(-\frac{\beta}{\gamma}\right)^{\frac{\nu+1}{2}}e^{\frac{\beta}{2\gamma}}, \tag{2.7}$$

and

$$\int_0^\infty x^{\nu-1}e^{\frac{\beta}{x}}dx = (-\beta)^{\nu}\Gamma(-\nu) \tag{2.8}$$

tantamount to changing β by $-\beta$ in (2.4). We have thus shown a rather interesting fact. Restriction of analytical extension (AE) of (2.1) equals AE of the restriction of that relation. This reconfirms that Gradshteyn and Rizhik's AE is indeed correct. Eq. (2.8) displays a cut at $\Re\beta > 0$. One can then choose $(-\beta)^\nu = e^{i\pi\nu}\beta^\nu$, $(-\beta)^\nu = e^{-i\pi\nu}\beta^\nu$, or $(-\beta)^\nu = \cos(\pi\nu)\beta^\nu$. We select the last possibility and obtain

$$\int_0^\infty x^{\nu-1}e^{\frac{\beta}{x}}dx = \cos(\pi\nu)\beta^{\nu}\Gamma(-\nu), \tag{2.9}$$

an important result that we will use in Section 3.