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Spin glass transition in a simple variant of the Ising model on multiplex networks

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HIGHLIGHTS

- Spin-glass phase transition in the Ising model on multiplex networks is studied.
- Critical temperature is evaluated using the replica method, for the replica-symmetric solution.
- The de Almeida–Thouless line is identified.
- The scaling exponents for the spin-glass transition are studied.
- Satisfactory agreement with Monte Carlo simulations is obtained.

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ABSTRACT

Multiplex networks consist of a fixed set of nodes connected by several sets of edges which are generated separately and correspond to different networks ("layers"). Here, the Ising model on multiplex networks with two layers is considered, with spins located in the nodes and edges corresponding to ferromagnetic or antiferromagnetic interactions between them. Critical temperatures for the spin glass and ferromagnetic transitions are evaluated for the layers in the form of random Erdös–Rényi graphs or heterogeneous scale-free networks using the replica method, from the replica symmetric solution. Stability of this solution is investigated and location of the de Almeida–Thouless line is determined. For the Ising model on multiplex networks with scale-free layers it is shown that the critical temperature is finite if the distributions of the degrees of nodes within both layers have a finite second moment, and that depending on the model parameters the transition can be to the ferromagnetic or spin glass phase. It is also shown that the correlation between the degrees of nodes within different layers significantly influences the critical temperatures and the phase diagram. The scaling behavior for the spin glass order parameter is determined and it is shown that for the Ising model on multiplex networks with scale-free layers the scaling exponents can depend on the distributions of the degrees of nodes within layers. The analytic results are partly confirmed by Monte Carlo simulations using the parallel tempering algorithm.

1. Introduction

In the last two decades rapid advancement in the theory and applications of complex networks has taken place related to the widespread recognition of their importance in social life, natural sciences and technology [1,2]. An important part of this trend was development of research on complex systems in which interactions among their constituent parts are

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determined by the underlying structure of complex networks [3,4]. In this context much effort was devoted to study the effect of the complex structure of interactions on the behavior of generic models of statistical physics exhibiting collective phenomena such as phase transitions. For example, ferromagnetic (FM) phase transition in the Ising model on complex, possibly heterogeneous networks was studied by means of various analytic [5–8] and numerical [9,10] methods. Also spin glass (SG) transition [11,12] in the Ising and related models on complex networks with quenched disorder of FM and antiferromagnetic (AFM) interactions was investigated using, e.g., variants of the replica method [13–16], effective field theory [17,18] and Monte Carlo (MC) simulations [19,20]. In connection with recent interest in even more complex structures ("networks of networks") much attention has been devoted to multiplex networks (MNs) which consist of a fixed set of nodes connected by various sets of edges called layers [21–23]. MNs naturally emerge in many social systems (e.g., transportation or communications networks), and interacting systems on such structures exhibit rich variety of collective behaviors and critical phenomena. For example, percolation transition [24–26], cascading failures [27], diffusion processes [28,29], epidemic spreading [30,31], etc., were studied on MNs. Also the FM transition in the Ising model [32] as well as diversity of first-order, second order and mixed-order transitions in a related Ashkin–Teller model [33] were investigated in the above-mentioned models with the structure of MNs.

As a natural extension of the above-mentioned research in this paper the SG transition is studied in the Ising model with the quenched disorder of the exchange interactions superimposed on the underlying structure of a MN. In Section 2 the Hamiltonian of the model is defined, with spins placed on a fixed set of nodes and with separately generated sets of edges (layers), with possibly different distributions of the degrees of nodes, corresponding to randomly assigned FM and AFM exchange interactions; the layers can have, e.g., the structure of random Erdös–Rényi (ER) graphs [34] or heterogeneous scale-free (SF) networks [35] and are generated from the so-called static model [36,37]. In Section 3 the thermodynamic properties of the above-mentioned model are investigated by means of the replica method [11,12]. The approach used here follows the study of the dilute SG model with infinite-range interactions [38–44] and is a direct generalization to the case of MNs of a procedure applied successfully to investigate the FM and SG transitions in the Ising model on random ER graphs [38], heterogeneous SF networks [15] and the FM transition in the Ising model on MNs [32]. In Section 4 the FM and SG transitions from the paramagnetic (PM) state are investigated in the above-mentioned model, the corresponding critical temperatures are evaluated from the replica symmetric (RS) solution and the effect of the distributions of the degrees of nodes within consecutive layers as well as the influence of the correlations between them on the phase diagram is emphasized. Besides, these analytic results are partly compared with MC simulations. In Section 5 stability of the RS solution is investigated and the boundary between the FM and a reentrant SG phase called a mixed (M) phase is determined, the so-called de Almeida–Thouless (AT) line [45,11,12]. This boundary also depends substantially on the correlation between the degrees of nodes within different layers. In Section 6 the critical exponents for the SG order parameter are determined for the Ising model on MNs with different distributions of the degrees of nodes within layers. Section 7 is devoted to summary and conclusions.

2. The model

2.1. The Hamiltonian

MNs consist of a fixed set of nodes connected by several sets of edges; the set of nodes with each set of edges forms a network which is called a layer of a MN [22,23]. In this paper only fully overlapping MNs are considered, with all N nodes belonging to all layers. In the following, for simplicity, MNs with N nodes and only two layers denoted as $G^{(A)}$, $G^{(B)}$ are considered. The layers (strictly speaking, the sets of edges of each layer) are generated separately, and, possibly, independently. As a result, multiple connections between nodes are not allowed within the same layer, but the same nodes can be connected by multiple edges belonging to different layers. The nodes $i = 1, 2, \ldots, N$ are characterized by their degrees $k_i^{(A)}, k_i^{(B)}$ within each layer, i.e., the number of edges attached to them within each layer. The, possibly heterogeneous, distributions of the degrees of nodes within each layer are denoted as $p_{k^{(A)}}, p_{k^{(B)}}$, and the mean degrees of nodes within each layer as $\langle k^{(A)}\rangle$, $\langle k^{(B)}\rangle$.

In the Ising model on a MN with two layers two-state spins $s_i = \pm 1$ are located in the nodes $i = 1, 2 \ldots N$ and edges within the layers $G^{(A)}$, $G^{(B)}$ connecting pairs of nodes i, j correspond to exchange interactions with integrals $J_{ij}^{(A)}, J_{ij}^{(B)}$, respectively. The exchange integrals are quenched random variables. It should be emphasized that in the model under study there is only one spin s_i located in each node which interacts with all its neighbors within all layers. The Hamiltonian of the model is

$$H = -\sum_{(i,j)\in G^{(A)}} J_{ij}^{(A)} s_i s_j - \sum_{(i,j)\in G^{(B)}} J_{ij}^{(B)} s_i s_j, \tag{1}$$

where the sums are over all edges belonging to the layer $G^{(A)}$ ($G^{(B)}$).

Following the studies of the dilute Ising SG models with infinite-range interactions on random ER graphs [38] and SF networks [15] in this paper it is assumed that the exchange integrals within each layer can assume only two values $J^{(A)}$ ($J^{(A)} > 0$) and $-J^{(A)}$ as well as $J^{(B)}$ ($J^{(B)} > 0$) and $-J^{(B)}$ which are assigned to the edges of the layer $G^{(A)}$ ($G^{(B)}$) with probability $r^{(A)}$ and $1 - r^{(A)}$ ($r^{(B)}$ and $1 - r^{(B)}$), respectively, and that these assignments are independent for the two layers. Thus the distributions of the exchange integrals within each layer $P_{r^{(A)}}\left(\left\{J_{ij}^{(A)}\right\}\right)$, $P_{r^{(B)}}\left(\left\{J_{ij}^{(B)}\right\}\right)$ are independent and have the form

$$P_{r^{(A)}}\left(\left\{J_{ij}^{(A)}\right\}\right) = \prod_{(i,j)\in G^{(A)}} \left[r^{(A)}\delta\left(J_{ij}^{(A)} - J^{(A)}\right) + \left(1 - r^{(A)}\right)\delta\left(J_{ij}^{(A)} + J^{(A)}\right)\right]$$