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Unitary–scaling decomposition and dissipative behaviour in finite-dimensional unital Lindblad dynamics

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A B S T R A C T

We investigate a decomposition of a unital Lindblad dynamical map of an open quantum system into two distinct types of mapping on the Hilbert–Schmidt space of quantum states. One component of the decomposed map corresponds to reversible behaviours, while the other to irreversible characteristics. For a finite dimensional system, we employ real vectors or Bloch representations and express a dynamical map on the state space as a real matrix acting on the representation. It is found that rotation and scaling transformations on the real vector space, obtained from the real-polar decomposition, form building blocks for the dynamical map. Consequently, the change of the linear entropy or purity, which indicates dissipative behaviours, depends only on the scaling part of the dynamical matrix. The rate of change of the entropy depends on the structure of the scaling part of the dynamical matrix, such as eigensubspace partitioning, and its relationship with the initial state. In particular, the linear entropy is expressed as a weighted sum of the exponential-decay functions in each scaling component, where the weight is equal to $|\vec{x}_k(\rho)|^2$ of the initial state ρ in the subspace. The dissipative behaviours and the partition of eigensubspaces in the decomposition are discussed and illustrated for qubit systems.

1. Introduction

A dynamical map to describe the evolution of a system between two given time epochs is one of the most versatile mathematical objects used in physics, especially in quantum physics [1,2]. In a close quantum system, such a dynamical map is described by a strongly continuous one-parameter unitary group on the operator space or on its dual as inspired by the celebrated work of von Neumann [1]. Despite the fact that this formulation sets the fundamental framework for dynamical analysis in quantum physics, it has been shown to be limited by many restrictions. For instances, the system may not inherit the closeness property; or it may not reach an equilibrium state in finite time; or the time homogeneity of the dynamics may not hold [3–7]. Consequently, many extensions have been proposed to model quantum systems such as open systems, or those in non-equilibrium or non-stationary regimes; see [2,8–13] for more details on the development of this subject.

It is commonly known that an open quantum system is not governed by the Schrödinger-type differential equation since the unitary evolution arising from such the equation cannot adequately explain irreversible behaviours of the dynamics. However, when the Markovian property is assumed (i.e. the dynamics are time homogeneous; see Section 4.2), the alternative formulations have been successfully achieved to sufficiently explain physical phenomena in open quantum

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systems. These include the Kossakowski–Gorini–Sudarshan formulation of the dynamical maps for finite dimensional open quantum systems [14]; the more-general Lindblad formulation which includes the dynamics on infinite dimensional systems [15]; and the Davies' or Nakajima–Zwanzig constructions which interpret the Markovian property as a result of a certain limit in the composite dynamics [2,16,17].

For open quantum systems, any dynamical map can be characterised into three categories: (i) the Liouville–von Neumann type, where the dynamics are precisely described by unitary groups [1]; (ii) the Lindblad type, where the complete positivity of a map and the Markovian property are required, while decay, decoherence or dephasing are allowed [15,18], and the dynamical maps can be treated as an ultra-weakly continuous one-parameter semigroup on the operator space; and (iii) the beyond-Lindblad type, where the assumptions on complete positivity or the Markovian property are relaxed [2,19]. Among these types of the dynamical maps, we observe that the most significant different characteristics is the entropy change with the dynamics [20,21]. In the Liouville–von Neumann type, the entropy change is zero by the unitary invariance, whereas in the Lindblad type, the entropy change essentially increases and possesses an asymptote, signalling a steady state in thermalisation and relaxation of the system [10,18,22].

Interestingly, the entropy change beyond the Lindblad dynamics remains open. This leads us to investigate the entropy change as the characterisation parameter of the quantum dynamics. Our interests lie in the Lindblad dynamics and beyond of open quantum systems. We hypothesise that the quantum dynamics should be characterised by two parameters, one for a reversible or coherence process, and the other for an irreversible or decoherence process. While this idea follows from intuition, it has not been explicitly verified or used to extract information about the evolution of an open quantum system. In particular, the relationship between the reversible and irreversible components of the process has never been explicitly derived on the level of a dynamical map, rather than that of a generator. This technique can be beneficial in case that the dynamical map does not obviously exhibit a generator, for example, a dynamical map derived from a process tomography.

A key idea of this work lies in a well-known polar decomposition of a matrix, which allows us to perform the unitary–scaling decomposition of a dynamical map. In a nutshell, we characterise the concerning dynamical matrices into two types: the rotation matrix describing the unitary or coherence behaviour, and the scaling matrix describing the dissipative or decoherence behaviour of the dynamics. As a general goal, we conjecture that such a decomposition should be valid in all three mentioned types of dynamics. However, for various technical difficulties (see Section 2.2 for more discussion), we have presented here the results for the unital Lindblad dynamics in a finite-dimensional open quantum system, where we obtain an exact relation between rotation and scaling components, as well as the effects of the initial state in the entropy change. In essence, the entropy change depends on not only the dynamical map and the initial state, but also the interplay relationship between them, i.e. the linear entropy will change with different rates if the initial state is prepare in different subspaces of the dynamical map. We show that this assertion is valid generally for a normal dynamical map.

The article is organised into sections as follow. Section 2 contains necessary background and relevant mathematical preliminaries of a dynamical map of an open quantum system. This section also includes the scope and the discussion of important assumptions for our current work. Based the polar decomposition of a matrix, the unitary–scaling scaling decomposition of a dynamical map is derived in Section 3. In Sections 4 and 5, the consequent results are presented, where in Section 4, we focus on the contribution to the dynamics in the isotropic scaling case. More importantly, the results on entropy change and characteristic of dissipative behaviours are presented in Section 5. The application of the decomposition to qubit systems are remarked in both Sections 4 and 5. We note that we illustrate the dissipative behaviour of the dynamics by using the linear entropy, which retrieves the characteristics of the Lindblad dynamics as expected. Finally the conclusions are summarised in Section 6. Additionally, relevant information and results on simple examples of elementary physical processes in qubit systems can be found in Appendices.

2. Mathematical preliminaries

2.1. Matrix and general Bloch representation

Finite-dimensional quantum systems can be commonly represented by real vectors (aka Bloch vector representation, or coherent representation) [23,24]. The state-observable description is given by a pair $(\rho, \mathbf{a})$ of a density matrix $\rho \in \mathsf{S}_d := \{\rho \in \mathsf{M}_d(\mathbb{C}) : \rho \geq 0, \mathrm{Tr}\,(\rho) = 1\}$, and an observable $\mathbf{a} \in \mathsf{H}_d := \{\mathbf{a} \in \mathsf{M}_d(\mathbb{C}) : \mathbf{a} = \mathbf{a}^*\}$ [25], where $\mathsf{M}_d(\mathbb{C})$ denotes the set of $d \times d$ complex matrices equipped with the Hilbert–Schmidt inner product $(\mathbf{a}, \mathbf{b})_{HS} := \mathrm{Tr}\,(\mathbf{a}^*\mathbf{b})$ and with norm $\|\mathbf{a}\|_{HS} := \sqrt{(\mathbf{a}, \mathbf{a})_{HS}}$. By choosing an orthogonal basis set $F = \{f_\alpha\}$ of H_d with $f_0 := \frac{1}{\sqrt{d}}\mathbf{I}_d$ and $\mathrm{Tr}\,(f_\alpha) := 0$, for $\alpha = 1, \ldots, d^2 - 1$, we can write a real vector representation for $\mathbf{a}$ in $\mathbb{R}^{d^2-1}$ as $\vec{x}(\mathbf{a}) := \big(x_1(\mathbf{a}), x_2(\mathbf{a}), \ldots, x_{d^2-1}(\mathbf{a})\big) \in \mathbb{R}^{d^2-1}$, where $x_\alpha(\mathbf{a}) := (\mathbf{a}, f_\alpha)_{HS} = \mathrm{Tr}\,(\mathbf{a}f_\alpha)$. We can therefore obtain

$$\mathbf{a} = \frac{\mathrm{Tr}\,(\mathbf{a})}{d}\mathbf{I}_d + \vec{f} \cdot \vec{x}(\mathbf{a}), \tag{1}$$

where $\vec{f} = \big(f_1, f_2, \ldots, f_{d^2-1}\big)$ is a $(d^2 - 1)$-tuplet of matrix bases f_α. For a density operator ρ, its real vector representation can be expressed as in Eq. (1) with $\mathrm{Tr}\,(\rho) = 1$.

Let Φ denote a dynamical map. This linear map can be represented as a matrix acting on the generalised Bloch vector. We assume some physical conditions on Φ: