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Effective dynamics of a random walker on a heterogeneous ring: Exact results

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ABSTRACT

In this paper, by considering a biased random walker hopping on a one-dimensional lattice with a ring geometry, we investigate the fluctuations of the speed of the random walker. We assume that the lattice is heterogeneous i.e. the hopping rate of the random walker between the first and the last lattice sites is different from the hopping rate of the random walker between the other links of the lattice. Assuming that the average speed of the random walker in the steady-state is v^*, we have been able to find the unconditional effective dynamics of the random walker where the absolute value of the average speed of the random walker is $-v^*$. Using a perturbative method in the large system-size limit, we have also been able to show that the effective hopping rates of the random walker near the defective link are highly site-dependent.

1. Introduction

Modeling of real systems by stochastic Markov process reveals many interesting aspects of the equilibrium and nonequilibrium properties of these systems. People are usually interested in the steady state properties of these processes, but also the fluctuations of physical quantities are quite important. A natural question is that how a rare event occurs and that what the stochastic dynamics which gives rise to this fluctuation is. It is known that this dynamics can be quite different from the dynamics of the original process in the long-time limit and that usually consists of nonlocal interactions even if the original process does not contain such interactions. An interesting example is the one studied in [1]. Considering a one-dimensional Glauber–Ising chain with its energy as a time-integrated physical observable. While it is a paramagnet in its equilibrium steady state at any finite temperature; however, by looking at the fluctuations of the dynamical observable it has been found that it might also show ferromagnetic or even antiferromagnetic phases. This justifies the importance of studying the fluctuations.

Many interesting, and yet important, physical observations can be obtained by studying the simplest models. They also give us intuitions for understanding our complex world. One of these models is a one-dimensional biased random walk. It is a nonequilibrium process which has been studied extensively in related literature. It has a vast applicability in different branches of science. In this paper we aim to study the fluctuations of the speed of a single random walker hopping on a one-dimensional lattice with a ring geometry in two different cases. In the first case we assume that the lattice is homogeneous i.e. the hopping rates between all of the lattice sites (or the links connecting the vertices of the network) are equal. In the second case we assume that one of the links is different from the other links of the lattice so that the random walker hops with a different hopping rate when it aims to pass this link in either directions. We call this the heterogeneous case. This model has already been studied from a different point of view. The authors in [2] and [3] have studied the heterogeneous

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case by considering entropy production rate of the random walker as an observable and found that the system undergoes a dynamical phase transition from a stable to a metastable phase.

Our main interest is to find an effective stochastic generator which generates the dynamics (sometimes called the effective dynamics) when the average speed of the random walker is atypical. This can be done using a generalized Doob's h-transform of a so-called tilted generator by working in a fluctuating ensemble [1,4,5]. Finding the effective dynamics associated with stochastic particle systems has recently received much attention (see [6] for a recent review).

In this paper we start with a brief review of the mathematical tools. By focusing on the heterogeneous case, we first calculate the probability distribution for the position of the random walker and also its average speed v^* in the steady state using a plane-wave method. The effective dynamics of the random walker at the point where its average speed is $-v^*$ has also been calculated exactly. It turns out that this dynamics is quite different from the simple dynamics of the random walker in the steady state. Using a perturbative approach in the large system-size limit we find that the system undergoes a dynamical phase transition. The effective dynamics of the system below this transition point is also obtained analytically.

This paper is organized as follows. In the first section we define the model. The necessary mathematical preliminaries are given in the second section. This will help non-experts to follow the paper much easier. We will then bring the results for the homogeneous and heterogeneous cases. The conclusion is the last section of this paper.

2. The model

Let us consider a stochastic process in which a classical particle hops on a one-dimensional closed lattice of length L. The dynamics of the particle is that of a biased random walker and basically a nonequilibrium one. Two different cases are considered. In the first case it is assumed that the lattice is homogeneous so that the links between consecutive lattice sites are similar. In this case the particle hops in the clockwise direction with transition rate x and in the reverse direction with the rate 1. In the second case it is assumed that the lattice is heterogeneous so that one of the links (say the link between the lattice sites L and 1) is different from the others. In this case the hopping rates between the links are similar to the first case except for the link between the lattice sites L and 1 where the particle hops with the hopping rate h in both directions between those two lattice sites. In order to study the fluctuations of the speed of the random walker in the above mentioned cases, we need some mathematical tools. Let us briefly review the most basic yet important concepts in order to present a self-contained manuscript.

3. Mathematical preliminaries

We define a probability vector $|P(t)\rangle = \sum_{n=1}^{L} P(n,t)|n\rangle$, with $\{|1\rangle, |2\rangle, \dots, |L\rangle\}$ as the basis vector associated with the position of the particle on the lattice. $P(n,t)$ is defined as the probability of being at the lattice site n at time t. The time evolution of the probability vector $|P(t)\rangle$ is governed by the following master equation

$$\frac{d}{dt}|P(t)\rangle = \hat{\mathcal{H}}|P(t)\rangle \tag{1}$$

where the time evolution is generated by a stochastic Hamiltonian $\hat{\mathcal{H}}$ whose matrix elements are

$$\langle n'|\hat{\mathcal{H}}|n\rangle = (1-\delta_{n,n'})\omega_{n\to n'} - \delta_{n,n'}\sum_{n''\neq n}\omega_{n\to n''} \quad \text{for} \quad n, n', n'' = 1, \dots, L \tag{2}$$

and that $\omega_{n\to n'}$ is the transition rate from configuration n to n' (the hopping rate of the random walker from the lattice site n to n' for our model). As we mentioned, we consider the speed of the random walker to be our physical observable as we aim to investigate the fluctuations of this quantity in certain cases. We first define $\mathcal{V}$ as a time-additive quantity with increments ± 1 ($+1$ for a clockwise hop and -1 for an anti-clockwise hop) which is a functional of the stochastic trajectory that the system follows during the observation time. A stochastic trajectory during the time interval $[t_0, t_f]$ is a sequence of M jumps in the configuration space

$$n(t_0) \to n(t_1) \to n(t_2) \to \cdots \to n(t_M)$$

taking place at time $t_1, t_2, \dots, t_M \in [t_0, t_f]$ where $n(t)$ means the location of the random walker at time t. Note that, the length of the time interval $T = t_f - t_0$ is given while the number of jumps M is a random variable that may take different values for different trajectories. The expectation value of the quantity $\mathcal{V}$ in the long-time limit reduces to $\langle \mathcal{V}\rangle = T\sum_{n,n'}\theta_{n\to n'}\omega_{n\to n'}P(n)$, i.e. it increases on average linearly with T. We expect that the corresponding probability distribution $P(\mathcal{V}/T = v)$ becomes more and more peaked around this value in the large-T limit. Assuming that the large deviation principle holds, the large deviation function of this probability distribution is defined by [7,8]

$$P(v) = \exp[-TI(v)] \tag{3}$$

in which $I(v)$ is called the rate function or the large deviation function. We have assumed that our observable satisfies the large deviation principle and that $I(v)$ is a convex rate function which can be obtained from the Legendre–Fenchel transform of the largest eigenvalue of a tilted generator defined as follows [8]

$$\langle n'|\hat{\mathcal{H}}(s)|n\rangle = (1-\delta_{n,n'})e^{-s\theta_{n\to n'}}\omega_{n\to n'} - \delta_{n,n'}\sum_{n''\neq n}\omega_{n\to n''} \quad \text{for} \quad n, n', n'' = 1, \dots, L \tag{4}$$