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Behavior of a stochastic SIR epidemic model with saturated incidence and vaccination rules

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HIGHLIGHTS

- The system has a unique global positive solution with any positive initial value.
- Random effect may lead to disease extinction under a simple condition.
- Sufficient condition for persistence has been established in the mean of the disease.

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ABSTRACT

In this paper, the threshold behavior of a susceptible–infected–recovered (SIR) epidemic model with stochastic perturbation is investigated. Firstly, it is obtained that the system has a unique global positive solution with any positive initial value. Random effect may lead to disease extinction under a simple condition. Subsequently, sufficient condition for persistence has been established in the mean of the disease. Finally, some numerical simulations are carried out to confirm the analytical results.

1. Introduction

The SIR epidemic model is one of the most important models in epidemiology and disease control. Kermack and McKendrick [1] initially proposed and investigated the classical SIR model. From then on, a lot of epidemic models are formulated as dynamical systems of ordinary differential equations [2–8]. The deterministic SIR model can be expressed by the following ordinary differential equations:

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \beta S(t)I(t) - \delta S(t) \\ \dot{I}(t) = \beta S(t)I(t) - (\mu + \gamma + \varepsilon)I(t) \\ \dot{R}(t) = \gamma I(t) - \mu R(t) + \delta S(t) \end{cases} \tag{1}$$

subject to $S(t) + I(t) + R(t) = N$ along with the initial values $S(0) = S_0 > 0$ and $I(0) = I_0 > 0$, where $S(t)$, $I(t)$ and $R(t)$ are the population fractions of susceptible, infective and removed at time t, respectively. Λ denotes a constant input members into the population, μ represents the natural death rate, β is the contact rate, ε denotes the death rate due to disease, γ denotes the recovery rate of the infective individuals. It can make the susceptible has the immunity of the epidemic if injected susceptible with the vaccine. The number of people for immunity is positive proportional to the infection with ratio δ at time t, $\delta S(t)$ represents the members who have been removed from susceptible to removed. In the

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system, the basic reputation number is $R_0 = \frac{\beta\Lambda}{(\mu+\gamma+\varepsilon)(\mu+\delta)}$. It always has a disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu+\delta}, 0, \frac{\delta\Lambda}{\mu(\mu+\delta)})$. When $R_0 \leq 1$, the disease-free equilibrium E_0 is global asymptotically stable, and therefore, the disease will die out after some period of time. When $R_0 > 1$, E_0 is unstable and there is an endemic equilibrium $E_* = (S^*, I^*, R^*)$, where $S^* = \frac{\Lambda}{(\mu+\delta)R_0}$, $I^* = \frac{\mu+\delta}{\beta}(R_0 - 1)$, $R^* = \frac{\gamma(\mu+\delta)}{\mu\beta}(R_0 - 1) + \frac{\delta\Lambda}{\mu(\mu+\delta)R_0}$, which is a global attractor, and so means the disease will prevail and persist. These conclusion of system (1) without vaccine can be found from [9].

In many previous epidemic models, the bilinear incidence rate βSI is frequently used [10–13]. To make system more interesting and realistic, it is reasonable to adopt the saturated incidence rather than bilinear incidence [14,15]. In this paper, the saturated incidence rate $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$ is adopted, so the model with saturated incidence takes the following form:

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - \delta S(t) \\ \dot{I}(t) = \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - (\mu+\gamma+\varepsilon)I(t) \\ \dot{R}(t) = \gamma I(t) - \mu R(t) + \delta S(t). \end{cases} \tag{2}$$

As a matter of fact, there are real benefits to be gained in using stochastic models because real life is full of randomness and stochasticity. As an extension of system (1), the random perturbation in model (2) is introduced by replacing the parameter β by $\beta + \sigma\dot{B}(t)$, where $\dot{B}(t)$ is the white noise, namely, $B(t)$ is a standard Brownian motion and σ denotes the intensity of the white noise. The model takes the following form:

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - \delta S(t) - \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t) \\ \dot{I}(t) = \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - (\mu+\gamma+\varepsilon)I(t) + \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t) \\ \dot{R}(t) = \gamma I(t) - \mu R(t) + \delta S(t). \end{cases} \tag{3}$$

Since the dynamic of R has no influence on the transmission dynamics, the third equation of system (3) can be omitted. The following system is investigated:

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - \delta S(t) - \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t) \\ \dot{I}(t) = \dfrac{\beta S(t)I(t)}{1+\alpha I(t)} - (\mu+\gamma+\varepsilon)I(t) + \dfrac{\sigma S(t)I(t)}{1+\alpha I(t)}dB(t). \end{cases} \tag{4}$$

The main aim of this paper is to investigate the dynamics of system (4).

The organization of this paper is as follows. In Section 2, the global existence and positivity of the solution to system (4) are investigated. In Section 3, it is shown that random effect may lead the disease to extinction under a simple condition. A sufficient condition for persistence in the mean of the disease is given in Section 4. Section 5 is devoted to introducing some numerical simulations to illustrate theoretical results. Finally, some conclusions are given.

2. Existence and uniqueness of the positive solution

Throughout this paper, let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets), and let $B(t)$ be a scalar Brownian motion defined on the probability space. As we have known, for any given initial value, the coefficients of a stochastic differential equation should satisfy the linear growth condition and the local Lipschitz condition to guarantee a unique global solution. It is easy to check that the coefficients of model (4) satisfy the local Lipschitz condition, while they do not satisfy the linear growth condition. Therefore, the solution of model (4) might explode within finite time. According to the analytical methods demonstrated in recent contributions [16–20], the existence and uniqueness of a global positive solution is obtained in the following Theorem 1.

Theorem 1. *For any initial value $(S(0), I(0)) \in R^2_+$, model (4) admits a unique global positive solution $(S(t), I(t))$ for all $t \geq 0$, and the solution remains in R^2_+ with probability one, namely*

$$P\{(S(t), I(t)) \in R^2_+, forall t \geq 0\} = 1.$$

Proof. Since the coefficients of system (4) are locally Lipschitz continuous, the model admits a unique local solution on $t \in [0, \tau_e)$ for any given initial value $(S(0), I(0)) \in R^2_+$, where τ_e is the explosion time [16–20]. To show this assertion, it need to show that $\tau_e = \infty$ a.s. Suppose that $\tau_k < +\infty$, let $k_0 > 0$ be sufficiently large such that either component of $(S(t), I(t))$ lies within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, the stopping time can be defined that

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{S(0), I(0)\} \leq \frac{1}{k}, or \max\{S(0), I(0)\} \geq k\}.$$