Physica A 501 (2018) 153–163

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Relationships between Perron–Frobenius eigenvalue and measurements of loops in networks

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HIGHLIGHTS

- The PFE have approximate power-law relationship with the number of loops in networks.
- The PFE have approximate power-law relationships with the average loop degree of nodes in networks.
- A majority of the loop ranks of networks obey Weibull distribution.
- The parameters set of Weibull distribution which loop ranks obey have approximate power-law relationships with the PFE.

ARTICLE INFO

Article history:
Received 8 July 2017
Received in revised form 4 January 2018
Available online 23 February 2018

Keywords:
Complex networks
Perron–Frobenius eigenvalue
Coefficient of network effect
Measurements of loops
Power-law

ABSTRACT

The Perron–Frobenius eigenvalue (PFE) is widely used as measurement of the number of loops in networks, but what exactly the relationship between the PFE and the number of loops in networks is has not been researched yet, is it strictly monotonically increasing? And what are the relationships between the PFE and other measurements of loops in networks? Such as the average loop degree of nodes, and the distribution of loop ranks. We make researches on these questions based on samples of ER random network, NW small-world network and BA scale-free network, and the results confirm that, both the number of loops in network and the average loop degree of nodes of all samples do increase with the increase of the PFE in general trend, but neither of them are strictly monotonically increasing, so the PFE is capable to be used as a rough estimative measurement of the number of loops in networks and the average loop degree of nodes. Furthermore, we find that a majority of the loop ranks of all samples obey Weibull distribution, of which the scale parameter *A* and the shape parameter *B* have approximate power-law relationships with the PFE of the samples.

1. Introduction

Almost all directed/undirected and unweighted networks can be represented as matrices of 1's and 0's, which are called adjacency matrices, for mathematical operations. A very rich and formal field of mathematics exists to perform these operations, one of the most useful operations is the calculation of eigenvalues. The Perron–Frobenius theorem states that, for matrices of 1's and 0's, there exists at least one real non-negative eigenvalue larger than all others, which is called the Perron–Frobenius eigenvalue (PFE). The PFE is an important parameter of networks which widely used in researches of graph theory [1], chaos [2] and complex networks [3–7]. Loops are subgraphs responsible for the multiplicity of paths going from

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https://doi.org/10.1016/j.physa.2018.02.180
0378-4371/

one to another generic node in a given network [8]. In 2004, J. R. Cares analyze a few samples of network and found that the PFE have some relationships with the number of loops in networks: (1) The PFE of network with the absence of a loop is 0; (2) The PFE of network with the presence of a simple loop is 1; (3) The more loops the network has, the bigger its PFE is. So he put forward his conjecture: the PFE can be used as a measurement of the number of loops in networks [9]. In 2005, J. R. Cares put forward the coefficient of network effect ($\mathrm{CNE} = \mathrm{PFE}/N$) as a measurement of the average number of loops each node participates in to avoid the influence of network scale N [10]. As counting the number of loops in network is much more difficult than calculating the PFE of it, Cares' two measurements are widely accepted [11–17], though his conjecture has not been proved yet.

What exactly is the relationship between the PFE and the number of loops in networks? Is it strictly monotonically increasing? What are the relationships between the PFE and other measurements of loops in networks? In this paper, we make our research on these three questions.

2. Measurements of loops

2.1. Definitions

The path starts and ends in the same node is called 'loop' in the network, in this paper, we define the 'loop' as a path satisfies the following sentences:

(1) It should starts and ends in the same node;
(2) It should pass through 3 nodes at least, including the start/end node;
(3) It should not pass through the same node or edge more than once.

There are many parameters which can be used as measurements of loops in network, such as the number of loops in network and the average loop degree of nodes used by J. R. Cares. In addition to these two parameters, we consider the distribution of loop ranks is another important measurement of loops in network.

(1) *The number of loops in network*. The number of loops in the network N_{loop} is an global measurement which indicates the networked effects of the whole network, it is not related with the scale or complexity of the network, large scale complex network may have no loop, such as the Bayesian network is Directed Acyclic Graph. It should be noted that, loops pass through the same nodes and edges but have different directions are count as two.
(2) *The average loop degree of nodes*. The loop degree of node $D_{loop}(v_i)$ is the number of loops node v_i participates in, it is an local measurement which indicates the node's networked effect. The average loop degree $\bar{D}_{loop} = \sum_{i=1}^{N} D_{loop}(v_i)/N$ is the average number of loop degree of all nodes in the network, it reflects the average networked effects of all nodes.
(3) *The distribution of loop ranks*. The loop rank is the sum of nodes/edges in a loop, its distribution $P_R(k)$ is the percentage of loops of which rank is k in the network. The distribution of loop ranks is an global measurement which describes the length of loops in the whole network, it should be represented by distribution types and distribution parameters set.

2.2. Algorithm

There exist a few researches on counting the number of loops/circles in a graph: Bianconi and his colleagues found the analytic expression for the scaling of number of loops of size h in the BA scale-free network [18–20,8,21], but his analytic expression is only applicable to the BA scale-free network; Noh gave out the expressions to evaluate the number of loops in correlated/uncorrelated scale-free network, the GN model and the assortative ERG model [22], but his evaluating expressions are only applicable to certain type of networks too; then, Bianconi and Gulbahce derived a Belief-Propagation algorithm for counting large loops in a directed network [23], but he noted that for the network with small number of nodes, the algorithm does not provide a good approximation for the number of loops present in the graph. Recently, loop is widely considered to be a special community structure of complex networks [24,25], there are a lot of community structure detection algorithms and quite a number of them do not limit the type of network. We explore some novel algorithms [26–30] and try to improve them so that they can be applied to search for loops in the network. However, by sample verification, it is found that some of these algorithms are not suitable for searching loop structure at all [26,27], some of these algorithms can find out a few but not all of the loops in the network [28,29], and some of these algorithms can find out all the loops in the network, but they are mixed with other common communities [30]. As yet (Nov. 2nd 2017), we have not found any algorithms in published references that can find out all the loops in the network exclusively with no limitation of the type of network.

In order to calculate the three measurements of loops defined in Section 2.1, we need an algorithm which is applicable to find out all the loops in any types of networks exhaustively and exclusively, the only input required is the adjacency matrix of the network, and its output should include all details of every loops in the network. Considering a closed loop can be divide into two parts: one directed edge from node v_j to node v_i, and one available path from node v_i to node v_j, so we transform the loop search problem to available path search problem. The Algorithm 1 below is used to find all available paths (loops