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Effective augmentation of networked systems and enhancing pinning controllability

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HIGHLIGHTS

- A perturbation theory is used to approximate influence of link addition in pinning controllability.
- A method is proposed to augment two disjoint networks while maximising the pinning control.
- The proposed method works much better than heuristic methods.

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ABSTRACT

Controlling dynamics of networked systems to a reference state, known as pinning control, has many applications in science and engineering. In this paper, we introduce a method for effective augmentation of networked systems, while also providing high levels of pinning controllability for the final augmented network. The problem is how to connect a sub-network to an already existing network such that the pinning controllability is maximised. We consider the eigenratio of the augmented Laplacian matrix as a pinning controllability metric, and use graph perturbation theory to approximate the influence of edge addition on the eigenratio. The proposed metric can be effectively used to find the inter-network links connecting the disjoint networks. Also, an efficient link rewiring approach is proposed to further optimise the pinning controllability of the augmented network. We provide numerical simulations on synthetic networks and show that the proposed method is more effective than heuristic ones.

1. Introduction

Many man-made and natural systems can be modelled as complex networked systems where a number of individual entities, known as agents, nodes or vertices, are coupled through connecting links, also known also as edges. *Network Science*, that study properties of real-world networks and develop proper models to capture their behaviour, is now a mature cross-disciplinary field of science with many applications in science and engineering [1,2]. Many real networked systems share some common structural properties such as heavy tailed power-law degree distribution, densification, small-worldness and community structure [3]. There has been overwhelming interest in studying dynamics on and of complex networks. "Dynamics of networks" refers to studying evolution of their structure, e.g. how the number of nodes/edges changes over the time. "Dynamics on networks" refers to studying collective behaviour of dynamical systems on networks.

Collective behaviour of a number of dynamical phenomena has been studied on complex networks. Examples include information cascade [4], cooperation [5], disease propagation and vaccination [6], consensus [7] and synchronisation [8,9]. Synchronisation is the most well-known collective behaviour studied in networked systems. When two or more dynamical

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systems interact through a network, they can synchronise their activity if the coupling strength between them is strong enough. Synchronisation properties of networks depend on three elements: the dynamics of individual nodes, the coupling strength and the structure of the connection graph. Node dynamics determines how the states evolve and the connection structure determines the topology of communication between the individual units. It is possible to show that under some conditions the network structure is the major contributor to the synchrony [10,11].

Identical dynamical systems coupled through a connected network can undergo self-organised synchrony, i.e. starting from different initial conditions, their trajectories asymptotically converge to the synchronisation manifold if the coupling strength between them is strong enough. However, in some applications one might need to control the dynamics of the network, e.g. to provider faster synchronisation [12]. Some applications require controlling dynamics of a networked system to a specific set point. Such a control strategy is referred to as pinning control (or pinning synchronisation) in the literature, where external input is fed into a (small) fraction of nodes to synchronise the dynamics of the whole network into a reference state [13–15]. The nodes to which the control signal is applied are called *driver nodes* and their proper selection has an important role on the properties of the pinning synchronisation. There are a number of approaches to measure pinning controllability of networks, i.e. the ease by which pinning synchronisation can be achieved in the network. Some approaches are based on Lyapunov method [16], and some others are based on the master stability function formalism [17]. Eigenratio of the augmented Laplacian matrix of the connection graph, i.e. the largest eigenvalue divided by the smallest one, is as effective metric measuring pinning controllability of networks [14,18,19]. This method is based on the assumption of having the same dynamics for the reference state and individual nodes.

Eigenratio has been frequently used as a controllability measure in the literature. It is based on the master stability function formalism [17] that obtains a condition that depends on the structure of the connection graph. This means that one can enhance the controllability by engineering the topology of the network. A number of methods have been proposed to enhance pinning controllability of networks. For example, one can employ link weighting [20] or efficient rewirings [21] to enhance the pinning controllability. Location of driver nodes has also a major role in pinning controllability of networks. Controlling a node involves significant cost, and one is often required to achieve the control task with minimum number of drivers, or the best drivers if their number is fixed. The pinning control with best drivers has been modelled as an optimisation problem and evolutionary optimisation methods have been used to solve the problem to obtain the list of best drivers [22]. Zhou et al. proposed to use techniques available in influence maximisation to choose optimal drivers [23]. Moradi Amani et al. proposed a simple metric based on the eigenvectors of the Laplacian matrix to find the most influential node for pinning control [24]. They found that controlling the hub nodes with the highest degree or betweenness centrality does not lead to the best control action in many cases.

In this paper, we propose an approach for effective augmentation of complex networks and enhance the pinning controllability conditions of the augmented network at the same time. Network augmentation has significant engineering applications [25,26]. For example, when a new suburb is added to the energy grid, the question is how to create the connections between the two networks provided that a certain utility function is maximised. Here, we consider the case when a new sub-network is added to an already existing network with known driver nodes. The question is how to create inter-network links between these two networks such that high level of pinning controllability is obtained in the final augmented network. Our proposed method is based on eigenvalue perturbation and we show that the proposed method has significantly better performance than heuristic methods such as connecting the hub nodes from each side.

2. Pinning controllability of complex dynamical networked systems

There are various forms of synchronisation such as complete, lag and phase synchronisation [27]. Here we consider only complete synchronisation between identical dynamical systems. Moreover, we assume that the reference state has the same dynamics as the individual nodes. Let us consider a network with N nodes, all having the same dynamics. First, let us consider the case when there is no pinning control where the individual dynamical systems spontaneously synchronise their activity through interacting via a connected network. The dynamics of the motion of coupled systems read

$$\frac{d\boldsymbol{x}_i}{dt} = F(\boldsymbol{x}_i) + \sigma \sum_{j=1}^{N} a_{ij} H\left(\boldsymbol{x}_i - \boldsymbol{x}_j\right); \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{x}_i$ are d-dimensional state vectors, F defines the identical dynamics of each node, σ is a uniform coupling strength and H is a projection matrix indicating from which dimensions the individual dynamical systems are coupled to one another. $A = (a_{ij})$ is the adjacency matrix of the connection graph; $a_{ij} = a_{ji} = 1$ when there is a link between nodes i and j, and $a_{ij} = a_{ji} = 0$ otherwise. We suppose that there are no self-loops in the network, i.e., $a_{ii} = 0$. Here we consider unweighted and undirected networks; however one can follow similar procedure for directed and/or weighted networks. One can rewrite the above equation as

$$\frac{d\boldsymbol{x}_i}{dt} = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} l_{ij} H\boldsymbol{x}_j; \quad i = 1, 2, \ldots, N, \tag{2}$$

where $L = (l_{ij})$ is Laplacian matrix of the connection graph; $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = k_i$, where k_i is degree of node i, which is calculated by summing all connections pointing to i. The Laplacian is a zero-raw sum matrix with positive diagonal elements, and such properties results in some interesting spectral properties for this matrix [28,29].