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The quotient of normal random variables and application to asset price fat tails

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HIGHLIGHTS

- Fat tails are shown to be a consequence of a quotient of normal distributions.
- Fat tails arise endogenously from the price equation of the supply/demand ratio.
- The density is shown to have behavior $f(x) \sim x^{-2}$ for large x.
- The coefficient of x^{-2} and smaller terms in the density are obtained.
- The exact density is calculated for anti-correlated numerator and denominator.

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ABSTRACT

The quotient of random variables with normal distributions is examined and proven to have power law decay, with density $f(x) \simeq f_0 x^{-2}$, with the coefficient depending on the means and variances of the numerator and denominator and their correlation. We also obtain the conditional probability densities for each of the four quadrants given by the signs of the numerator and denominator for arbitrary correlation $\rho \in [-1, 1)$. For $\rho = -1$ we obtain a particularly simple closed form solution for all $x \in \mathbb{R}$. The results are applied to a basic issue in economics and finance, namely the density of relative price changes. Classical finance stipulates a normal distribution of relative price changes, though empirical studies suggest a power law at the tail end. By considering the supply and demand in a basic price change model, we prove that the relative price change has density that decays with an x^{-2} power law. Various parameter limits are established.

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1. Introduction

A long-standing puzzle in economics and finance has been the "fat tails" phenomenon in relative asset price change and other observations that refers to the empirically observed power-law decay rather than the expected classical exponential. In practical terms, this means unusual events are less rare than expected, resulting in broad implications as discussed below. Given the quite general application of the Central Limit Theorem, it is natural to expect that the frequency of the relative price change as a function of the relative price change would result in a normal, or Gaussian, distribution with exponential decay, i.e., the density has the same tail as $f(x) = \left(2\pi\sigma^2\right)^{-1/2} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$ where μ is the mean and σ^2 is the variance.

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The assumption of a normal distribution for relative price changes dates back to Bachelier's thesis [1] in 1900, and was further popularized in the Black–Scholes work on options pricing [2]. Champagnat et al. [3] note several reasons for the saliency of utilizing log-normal price changes. First, they can be "simply interpreted and estimated. Second, closed-form expression exists for several options. Third, they could be embedded in a continuous time process, as the geometric Brownian motion, which models the evolution of the stock over the time. Indeed, many theories, for example the Capital Asset Pricing Model (CAPM) for portfolio management, take their roots in the Gaussian world".

One particular application involves "Value-at-Risk" which addresses questions such as: Can we expect that the investment will retain at least, say, 75% of its current value within a five year time period with a 95% confidence? This methodology is often at the heart of risk analysis of an investment portfolio. The Gaussian assumption facilitates calculations; however, there have been numerous studies that indicate the risk is understated as a result [3–5]. An early study by Fama [6] provided empirical evidence that there were ten times as many observations of relative price changes than would be expected at four standard deviations from the classical theories stipulating normal distributions. Mandelbrot and Hudson [7] express their perspective in a section heading: "Markets are very, very risky — more risky than the standard theories imagine". In a more generalized context, Taleb [8,9] has long asserted that unusual events occur far more often than one would expect from the normal distribution. Another aspect of work in this area has involved modeling [10,11]. The empirical observations of fat tails have also been noted in high frequency trading [12].

Thus, the question of the tail of the distribution of relative price changes is important in several key areas of finance and economics. As a practical matter it would appear that one can investigate empirically, and implement the conclusions without reference to a particular model. While data seems to be abundant, the problem is that obtaining a large amount of it often entails using data from older time periods that may well be irrelevant. Hence, the theoretical examination of the origins of the tail of the distribution becomes crucial. Various explanations have been offered for the observation of fat tails. These differ from our approach in that they stipulate an exogenous influence that alters the natural normal decay. For example, it has been argued that large institutional investors placing trades in less liquid markets tend to cause large spikes [13]. Theoretical models, e.g., [11], using random walk have been used to explain fat tails. We refer to the book by Kemp [14] for a review of the literature. While these may be important factors that lead to fat tails, we demonstrate below that fat tails also arise from the endogenous price formation process.

In the classical approach to finance the basic starting point is the stochastic differential equation for the relative price change, $P^{-1}dP = d\log P$ as a function of time, t, and $\omega \in \Omega$ (with Ω as the sample space):

$$d\log P = \mu dt + \sigma dW. \tag{1.1}$$

Here W is the standard Brownian motion, so $\Delta W := W(t) - W(t-\Delta t) \sim \mathcal{N}(0, \Delta t)$, i.e., W is normal with the variance as Δt and mean 0, and has independent increments. The stochastic differential equation above is shorthand for the integral form (suppressing ω in notation)

$$\log P(t_2) - \log P(t_1) = \int_{t_1}^{t_2} \mu dt + \int_{t_1}^{t_2} \sigma dW \tag{1.2}$$

where μ and σ can be constant, functions of t, or random variables. For μ, σ constant, and $\Delta t := t_2 - t_1$, one can write

$$\Delta \log P := \log P(t_2) - \log P(t_1) = \mu \Delta t + \sigma \Delta W. \tag{1.3}$$

With the assumption that σ is nearly constant over time, classical finance clearly stipulates that the relative temporal changes in asset prices should be normal. The basic equation (1.1) is obtained from the idea that all information about the asset is incorporated into the price, and that random world events alter the value on each time interval. Further, the assumption of the existence of a vast arbitrage capital means that the changes in the valuation are immediately reflected in the price, notwithstanding any bias or mistake on the part of the less knowledgeable investors. Consistent with the Central Limit Theorem, it is assumed that the events that alter the asset valuation are normally distributed. But the important and tacit second assumption is that relative price changes inherit this property.

While (1.1) is the basis for a large majority of papers on asset prices and related issues in finance, it is difficult to generalize in some directions. Indeed, with the assumption of infinite arbitrage already built into the model, what can one subtract from infinity in order to obtain the randomness that arises from the finiteness of trader assets and order flow? What is needed then is an approach that takes into account more of the microstructure of trading, i.e., the supply and demand of the asset submitted to the market clearinghouse (see e.g., [15–17] and references therein).

In this paper we examine the temporal evolution in percentage price changes by modeling price change that utilizes a fundamental approach of supply/demand economics analysis. We show that if one assumes supply and demand are normally distributed, the mathematics of the quotient of normals suffices to yield fat tails. In particular, in the limit of large deviations from the mean, one obtains the result $f(x) \simeq f_0 x^{-2}$ for the density for large x, where the constant f_0 depends on the means, variances and correlations of supply and demand.

Thus modeling of the relative price change in terms of finite supply and demand lead naturally to the basic statistical problem of the distribution of the quotient of two normal random variables. While there is a long history of the problem, surveyed below, most results concern the mid-range of the distribution, rather than the tail. We obtain a number of exact representations and rigorous bounds on the density conditioned upon the signs of the numerator and denominator, as well as the overall density for all correlations ρ such that $|\rho| < 1$. For $\rho = -1$ we obtain a particularly simple exact expression for the density for all $x \in \mathbb{R}$.

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