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Characterization of time series via Rényi complexity–entropy curves

M. Jauregui [a], L. Zunino [b,c], E.K. Lenzi [d], R.S. Mendes [a], H.V. Ribeiro [a,*]

[a] Departamento de Física, Universidade Estadual de Maringá, Maringá, PR 87020-900, Brazil
[b] Centro de Investigaciones Ópticas (CONICET La Plata - CIC), C.C. 3, 1897 Gonnet, Argentina
[c] Departamento de Ciencias Básicas, Facultad de Ingeniería, Universidad Nacional de La Plata (UNLP), 1900 La Plata, Argentina
[d] Departamento de Física, Universidade Estadual de Ponta Grossa, Ponta Grossa, PR 84030-900, Brazil

HIGHLIGHTS

- A new method for characterizing time series is proposed.
- The method is based on the Rényi entropy.
- The approach is useful for distinguishing noise from chaos.

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ABSTRACT

One of the most useful tools for distinguishing between chaotic and stochastic time series is the so-called complexity–entropy causality plane. This diagram involves two complexity measures: the Shannon entropy and the statistical complexity. Recently, this idea has been generalized by considering the Tsallis monoparametric generalization of the Shannon entropy, yielding complexity–entropy curves. These curves have proven to enhance the discrimination among different time series related to stochastic and chaotic processes of numerical and experimental nature. Here we further explore these complexity–entropy curves in the context of the Rényi entropy, which is another monoparametric generalization of the Shannon entropy. By combining the Rényi entropy with the proper generalization of the statistical complexity, we associate a parametric curve (the Rényi complexity–entropy curve) with a given time series. We explore this approach in a series of numerical and experimental applications, demonstrating the usefulness of this new technique for time series analysis. We show that the Rényi complexity–entropy curves enable the differentiation among time series of chaotic, stochastic, and periodic nature. In particular, time series of stochastic nature are associated with curves displaying positive curvature in a neighborhood of their initial points, whereas curves related to chaotic phenomena have a negative curvature; finally, periodic time series are represented by vertical straight lines.

1. Introduction

Quantifying the degree of complexity of a system is a common task when studying the most diverse complex systems. This task usually starts by constructing a time series and then considering a complexity measure. Since there is an inherent difficulty in defining the concept of complexity, researchers have employed several approaches/theories as possible

* Corresponding author.
E-mail address: hvr@dfi.uem.br (H.V. Ribeiro).

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complexity measures. A non-exhaustive list includes entropies [1], relative entropies [2], algorithmic complexities [3], fractal dimensions [4], Lyapunov exponents [5], and other traditional nonlinear time series methods [6]. However, most of these approaches suffer from the drawbacks of being strongly sensitive on tuning parameters, hindering the reproducibility of results.

A possible way to overcome these difficulties is to employ the permutation entropy, introduced by Bandt and Pompe [7]. This complexity measure is basically the Shannon entropy of the distribution of the permutations associated with d-dimensional partitions $(x_k, x_{k+1}, \dots, x_{k+d-1})$ of a time series $(x_1, \dots, x_m)$. It is common to choose $d \in \{3, 4, 5, 6, 7\}$ in most practical applications, in such a way that the number of permutations $d!$ is much smaller than m. For this reason, the computational cost of computing the permutation entropy is usually lower than the ones related to other complexity measures. Also, the idea of associating permutations with finite-dimensional partitions of a time series allows the application of this method to time series of arbitrary nature. These remarks agree with the fact that the method of Bandt and Pompe is already widely spread over the scientific community [8–17].

In some cases, the permutation entropy can distinguish among time series of regular, chaotic and stochastic behavior. However, Rosso et al. [18] have demonstrated that this complexity measure alone is not enough for properly performing this task. For instance, they have shown that time series related to the logistic map at fully developed chaos and time series associated with power-law correlated noises can display practically the same value of permutation entropy. Mainly because of that, Rosso et al. have employed the joint use of the permutation entropy and another complexity measure, called the statistical complexity [19–21]. The statistical complexity is basically the product of the permutation entropy by a distance between the distribution of the permutations and the uniform distribution. Having the values of the permutation entropy H and the statistical complexity C associated with a given time series, Rosso et al. have represented this series by a point (H, C) in a diagram of C versus H. This diagram is the so-called complexity–entropy causality plane, where the term causality refer to the fact that temporal correlations are taken into account by the Bandt and Pompe approach. In this representation space, time series of chaotic and stochastic nature are represented by points located in different regions, that is, noise and chaos can be distinguished by using the complexity–entropy causality plane.

However, we have recently depicted several situations in which the values of H and C are not enough for distinguishing among time series of distinct nature [22]; for instance, the points (H, C) can become very close to each other for time series displaying different periodic and chaotic behaviors. Motivated by this fact, we have extended the causality plane for considering the Tsallis [23] monoparametric generalization of the Shannon entropy [22]. The values of the parameter of the Tsallis entropy give different weights to the probabilities associated with the permutations; consequently, different dynamical scales of the system are accessed by varying the entropy parameter. In that article, we associated parametric curves with time series based on the different values of (H, C) obtained by changing the Tsallis entropy parameter, a representation that we call the complexity–entropy curve. These curves have proven to enhance the differentiation of time series of regular, chaotic and stochastic nature even in cases in which the usual complexity–entropy causality plane does not provide useful information.

On the other hand and similarly to what happens with the concept of complexity, there are several other entropy definitions in the context of information theory [24–26]. These different entropies allow us to explore, capture and quantify different forms of complexity, leading us to more suitable descriptions for characterizing the most diverse complex systems addressed by physicists. Here we further explore this idea by considering the Rényi entropy [27] in place of the Tsallis entropy [23]. The Rényi entropy is also a monoparametric generalization of the Shannon entropy, which has been employed in several contexts such as medical/diagnostics applications [28], time–frequency analysis [29], quantum entanglement measures [30,31], and image thresholding [32]. Therefore, in analogy to the Tsallis entropy case, we shall associate a parametric curve with a given time series (the *Rényi complexity–entropy curve*), and by exploring some properties of this curve, we can characterize the time series under study. Among other findings, we show that the curvature of these curves identifies whether a time series is of a stochastic or a chaotic nature, and that periodic time series are represented by vertical straight lines.

The organization of the article is as follows. Section 2 provides the definitions of the Rényi entropy and the Rényi statistical complexity. Section 3 gives a brief description of the method of Bandt and Pompe for defining the ordinal probabilities from a given time series. We also work out a list of general properties of the Rényi complexity–entropy curves. In Section 4, we analyze several time series of chaotic and stochastic nature, obtained by numerical procedures or by experimental measurements. Finally, we conclude in Section 5.

2. A definition of a statistical complexity based on the Rényi entropy

The Rényi entropy of a discrete probability distribution $p = (p_1, \dots, p_n)$ is defined as [27]

$$S_\alpha(p) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^\alpha \,, \quad \alpha > 0 \,,\ \alpha \neq 1 \,. \tag{1}$$

From this definition, we immediately note that $S_\alpha(p)$ recovers the Shannon entropy of p when α tends to 1. We can further verify that the maximum value of the Rényi entropy is equal to $\ln n$ (as in the Shannon entropy case), which happens when the uniform distribution $u = (1/n, \dots, 1/n)$ is considered. This enables us to define the normalized Rényi entropy of p as

$$H_\alpha(p) = \frac{S_\alpha(p)}{\ln n} \,. \tag{2}$$