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Time irreversibility from symplectic non-squeezing

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ABSTRACT

The issue of how time reversible microscopic dynamics gives rise to macroscopic irreversible processes has been a recurrent issue in Physics since the time of Boltzmann whose ideas shaped, and essentially resolved, such an apparent contradiction. Following Boltzmann's spirit and ideas, but employing Gibbs's approach, we advance the view that macroscopic irreversibility of Hamiltonian systems of many degrees of freedom can be also seen as a result of the symplectic non-squeezing theorem.

1. Introduction

There is little doubt that Gromov's symplectic non-squeezing theorem [1] is one of the most central results in Symplectic Geometry. This result is widely credited for concretely differentiating between symplectic and volume-preserving maps and, as a result, for establishing symplectic topology [2,3] as an independent and free-standing line of research. The method of pseudo holomorphic curves [4] that Gromov used to prove the non-squeezing theorem had a tremendous impact in several branches of Mathematics, as well as in String Theory, effects which are felt even today more than three decades after establishing that fundamental result.

Despite all this substantial impact in Mathematics, and String Theory, not many of its potential implications for Physics, and Statistical Physics in particular, have been explored, so far we know. A major exception that we are familiar with, are the works of M. de Gosson and his collaborators [5–10]. His papers on the non-squeezing theorem have made accessible, to a typical Physics audience, the fundamental ideas contained in Gromov's and the subsequent works on symplectic capacities [2].

In this work, we present a hand-waving argument purporting to show that one can ascribe the macroscopic time irreversibility of systems of many degrees of freedom, whose microscopic dynamics is given by a Hamiltonian evolution, to the validity of the non-squeezing theorem. We largely follow Boltzmann's fundamental ideas on this issue, but use Gibbs's entropy, employing a few of the more recent results on the symplectic non-squeezing theorem and the symplectic capacities on this matter.

In Section 2, we provide a few preliminaries about the symplectic non-squeezing theorem and symplectic capacities. Section 3 contains the main part of our argument. Section 4 provides some conclusions and speculations.

2. The non-squeezing theorem and symplectic capacities

2.1. Preliminaries

Consider the $2n$-dimensional symplectic manifold $(\mathcal{M}, \omega)$. We recall that for $x \in \mathcal{M}$ and vectors $X, Y \in T_x\mathcal{M}$, ω is a non-degenerate 2-form

$$\omega_x(X, Y) = 0, \quad \forall\, Y \in T_x\mathcal{M} \implies X = 0 \tag{1}$$

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which is also closed

$$d\omega = 0. \tag{2}$$

Let $\mathfrak{L}_X$ stand for the Lie derivative and i_X for the contraction along X. Then Cartan's formula gives

$$\mathfrak{L}_X\omega = d(i_X\omega) + i_X(d\omega) \tag{3}$$

which due to (2) reduces to

$$\mathfrak{L}_X\omega = d(i_X\omega). \tag{4}$$

Consider moreover $X_{\mathcal{H}}$ to be a Hamiltonian vector field: then, by definition $X_{\mathcal{H}} \in T\mathcal{M}$ is the generator of a Hamiltonian evolution/flow whose corresponding Hamiltonian function is $\mathcal{H} : \mathcal{M} \to \mathbb{R}$, where $X_{\mathcal{H}}$ is defined by

$$i_{X_{\mathcal{H}}}\omega = -d\mathcal{H}. \tag{5}$$

Substituting (5) into (4) we see that Hamiltonian vector fields preserve the symplectic form

$$\mathfrak{L}_{X_{\mathcal{H}}}\omega = 0. \tag{6}$$

This relation can be considered as a justification for the adoption of (2) in the definition of the symplectic form. The $2n$-form $\omega^n/n!$ is non-degenerate, so it can be used as the volume form of $(\mathcal{M}, \omega)$. Then

$$\mathfrak{L}_{X_{\mathcal{H}}}\left(\frac{\omega^n}{n!}\right) = 0 \tag{7}$$

which is Liouville's theorem on the preservation of the symplectic volume under Hamiltonian flows. We see that the invariance of ω under Hamiltonian flows implies Liouville's theorem. Motivated by this realization, the question that arose was whether there was actually any difference between symplectic and volume preserving diffeomorphisms of symplectic manifolds. The symplectic non-squeezing theorem [1] provided an affirmative answer.

A symplectic diffeomorphism $\phi : (\mathcal{M}, \omega) \to (\mathcal{M}, \omega)$ is a diffeomorphism preserving the symplectic structure

$$\phi^*\omega = \omega. \tag{8}$$

Let ω_0 indicate the standard symplectic form on $\mathbb{R}^{2n}$. Consider a local coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ where, in the Hamiltonian Mechanics terminology, y_i is the canonically conjugate momentum to x_i for $i = 1, \ldots, n$. Then

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{9}$$

By considering $(\mathbb{R}^{2n}, \omega_0)$ instead of a general symplectic manifold $(\mathcal{M}, \omega)$, one does not lose any generality since, by Darboux's theorem, all symplectic manifolds of dimension $2n$ are locally symplectically diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$ [2,3]. Hence, and in stark contrast to the Riemannian case, all symplectic manifolds are locally "equivalent": they can only be distinguished from each other by their dimension and by global invariants.

2.2. The symplectic non-squeezing theorem

In the terminology of the previous subsection, consider the open ball of radius r in $\mathbb{R}^{2n}$ endowed with the Euclidean metric:

$$B_{2n}(r) = \{\mathbb{R}^{2n} \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) : \quad x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 \leq r^2\}. \tag{10}$$

Let $Z_{(x_1,y_1)}(R)$ be a cylinder of radius R in $\mathbb{R}^{2n}$ based on the symplectic 2-plane (x_1, y_1):

$$Z_{(x_1,y_1)}(R) = \{\mathbb{R}^{2n} \ni (x_1, \ldots x_n, y_1, \ldots, y_n) : \quad x_1^2 + y_1^2 \leq R^2\}. \tag{11}$$

The symplectic non-squeezing theorem [1] is the statement that $B_{2n}(r)$ can be embedded by a symplectic diffeomorphism in any cylinder based on a symplectic 2-plane, such as $Z_{(x_1,y_1)}(R)$, if $r \leq R$. This is a non-obvious constraint and exists despite the fact that

$$vol(B_{2n}(r)) < vol(Z_{(x_1,y_1)}(R)) \tag{12}$$

the latter volume being, obviously, infinite.

Contrast this with the behavior of the volume-preserving maps: consider two compact domains $\Omega_1, \ \Omega_2 \subset \mathbb{R}^{2n}$ which are diffeomorphic, have smooth boundaries and equal volumes. Then [11] there is a volume-preserving diffeomorphism $\psi : \Omega_1 \to \Omega_2$. Therefore, the additional obstruction provided by the symplectic non-squeezing theorem to symplectic