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Gaussian random bridges and a geometric model for information equilibrium

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HIGHLIGHTS

- A new class of stochastic processes called GRBs are introduced and studied.
- Anticipative and non-anticipative dynamical models of GRBs are proved.
- GRBs are used as information processes on partially observed systems.
- Information equilibrium and diversity are studied using differential geometry.
- Higher dimensional information networks are discussed on a given topological space.

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ABSTRACT

The paper introduces a class of conditioned stochastic processes that we call Gaussian random bridges (GRBs) and proves some of their properties. Due to the anticipative representation of any GRB as the sum of a random variable and a Gaussian $(T, 0)$-bridge, GRBs can model noisy information processes in partially observed systems. In this spirit, we propose an asset pricing model with respect to what we call information equilibrium in a market with multiple sources of information. The idea is to work on a topological manifold endowed with a metric that enables us to systematically determine an equilibrium point of a stochastic system that can be represented by multiple points on that manifold at each fixed time. In doing so, we formulate GRB-based information diversity over a Riemannian manifold and show that it is pinned to zero over the boundary determined by Dirac measures. We then define an influence factor that controls the dominance of an information source in determining the best estimate of a signal in the $\mathcal{L}^2$-sense. When there are two sources, this allows us to construct information equilibrium as a functional of a geodesic-valued stochastic process, which is driven by an equilibrium convergence rate representing the signal-to-noise ratio. This leads us to derive price dynamics under what can be considered as an equilibrium probability measure. We also provide a semimartingale representation of Markovian GRBs associated with Gaussian martingales and a non-anticipative representation of fractional Brownian random bridges that can incorporate degrees of information coupling in a given system via the Hurst exponent.

1. Introduction

Many observed stochastic dynamic systems exhibit the processing of complex information. Such systems appear in numerous fields of scientific research including mathematical physics, biology, neuroscience and finance, where information

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is not necessarily treated purely as physical data, but also as an abstract mathematical object to help understand and model the phenomenon under study. In numerous cases, information can only be partially observed and state estimations need to be formulated. In such situations, one may further encounter multiple sources of information, especially in social systems, where the agents may be exposed to different observations and exhibit diverse inference for decision making.

The main goal of this paper is twofold: (i) to introduce and analyze a class of stochastic processes that can naturally model information dynamics in partially observed systems, (ii) to apply these processes in proposing an asset pricing model with respect to what we call information equilibrium arising in financial markets that exhibit information diversity due to multiple sources of information. The aforementioned stochastic processes, which we shall name Gaussian random bridges, can be applied in many different fields, and we provide several key results to allow this. On the other hand, information diversity typically appears in economic systems, where agents make decisions based on different information.

We adopt the term *information diversity* over *information asymmetry*, since the economics literature conventionally describes information asymmetry as situations where one agent has informational advantage over another. In this work we take a broader view to include situations where agents may be exposed to multiple different sources of information; some potentially less dominant than others in influencing the decisions and beliefs of agents. Therefore, information diversity would encompass scenarios which not only address the *amount* of information, but may also address the *quality* of information. We shall still highlight that information asymmetry has attracted considerable attention from mathematicians, and amongst numerous examples, [1–12] are only few names to mention, where some rely on the seminal work of [13]. Hence, the academic research on information asymmetry employed and gave birth to important results and ideas in applied probability.

We begin our study of partially observed systems by placing it in an avenue of stochastic processes defined over a finite time horizon that can be viewed as noisy information processes on a signal. In doing so, we introduce an extended class of conditioned processes that we call Gaussian random bridges (GRBs), which admit an anticipative representation as the sum of a random variable (e.g. a signal) and a Gaussian $(T, 0)$-bridge (e.g. noise). By definition (see Definition 2.1), the class of GRBs is an extension of Gaussian bridges introduced in [14] and the Brownian bridge information process of [15,16]. We prove various properties of GRBs. For example, if the Gaussian process associated to the GRB has stationary increments, then the increment process of a GRB is a space-shifted, time-shifted GRB. This result is analogous to the dynamic consistency property of the Brownian bridge information process of [16] and Lévy random bridges of [17,18]. In addition, if the underlying Gaussian process is Markov, then the corresponding GRB is also Markov. In the case of a Gaussian martingale with increasing brackets, the Markovian GRB further admits a semimartingale representation that paves its way to the universe of Ito calculus. As a non-Markovian, non-semimartingale example, we introduce fractional Brownian random bridges (FBRBs) and provide their non-anticipative representation. FBRBs can be used to incorporate memory to information flow, which we shall sometimes refer to as information coupling, in a given system. In addition, we prove a rather involved non-anticipative representation of any continuous GRB as an extension to [14]. We shall note that GRBs fit naturally to the quantum state reduction models of [15,19,20] and the information-based asset pricing framework of [16,21,22]. For example, in [15], the Brownian bridge information process is proposed to solve a stochastic Schrödinger equation. In [21], multiple sources of information are incorporated in a financial system and the so-called effective information process is derived in the context of information asymmetry. GRBs can readily extend both frameworks.

We shall elaborate more on our application of GRBs in deriving asset price dynamics in a system with multiple sources of information. Albeit information diversity constitutes an integral part of our application, the paper parts slightly in terms of the problem it puts forward: In a given system where multiple sources of information exist, can we propose a reasonable framework to systematically construct what may be called *information equilibrium*? That is, suppose a group of agents have access to noisy observations emerging from several information sources on some financial payoff to be realized at some maturity. It would be rational to assume that the opinions of these agents are formed depending on the influence of these information sources over the market, which may be driven by numerous factors such as the credibility of the information source, the diffusion power due to wider coverage, or lower cost restrictions. We postulate that the interactions of these agents produce an equilibrium filtration that defines a probability measure which amalgamates different beliefs to form price dynamics. Our aim is to propose a mathematical framework that models this filtration explicitly through what we call information equilibrium and deriving the corresponding price dynamics through a discounted conditional expectation process.

In our attempt to model information equilibrium, it turns out that differential geometry proves to be very useful, since it provides a mathematically rich structure that sheds light on sophisticated relations which may be hidden from a purely probabilistic point of view. Let us assume there are two sources of information in a given market. If we can naturally represent the flow of information as two distinct points moving in time on some metric space, then it becomes reasonable to characterize the information diversity as the distance between these two points on that space at each time. We can then aim to determine an equilibrium point on the same space at every fixed time as a function of information diversity. For example, the equilibrium can be formulated as the solution of an optimization problem that minimizes the difference of information diversities under various constraints, or it can be governed by time-dependent influence of information sources over agents' decision making. In summary, our idea is to project information on a topological manifold endowed with a metric that enables us to systematically determine an equilibrium point of a stochastic system that can be represented by multiple points on that manifold. In our attempt, we choose to define an information influence factor that controls the dominance of an information source in determining the best estimate of a signal in the $\mathcal{L}^2$-sense. This factor determines a