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Approximate solution of space and time fractional higher order phase field equation

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HIGHLIGHTS

- New fractional nonlinear partial differential equation has been proposed with Riesz in space and Caputo in time.
- The proposed equation give a generalization for higher order phase field equations, specifically the sixth order diffusion ones.
- The optimal homotopy analysis method (OHAM) with a suggested residual error function is successfully applied.
- The suggested fractional phase field equation supports both monotonic and oscillatory phase front solutions and this is justified via numerical examples.

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ABSTRACT

This paper is concerned with a class of space and time fractional partial differential equation (STFDE) with Riesz derivative in space and Caputo in time. The proposed STFDE is considered as a generalization of a sixth-order partial phase field equation. We describe the application of the optimal homotopy analysis method (OHAM) to obtain an approximate solution for the suggested fractional initial value problem. An averaged-squared residual error function is defined and used to determine the optimal convergence control parameter. Two numerical examples are studied, considering periodic and non-periodic initial conditions, to justify the efficiency and the accuracy of the adopted iterative approach. The dependence of the solution on the order of the fractional derivative in space and time and model parameters is investigated.

1. Introduction

In reaction–diffusion systems, a competition between the diffusion process and the reaction kinetics process occurs. As a result, reaction–diffusion equations have a nontrivial fruitful characteristics and play an indispensable rule in describing real phenomena, such as pattern formation and phase field transition, in physics, chemistry and biology.

The space–time fractional diffusion equation (STFDE) has been arisen from replacing the standard space and time partial derivative with a space and time fractional partial derivative. STFDE are incorporated into the research of various practical problems in applied sciences because fractional order derivatives are successfully used in modeling the memory and hereditary properties of different substances and composite materials. STFDE has been found to be of quite superior in modeling the anomalous diffusion; the processes associated with sub-diffusion and super-diffusion or both [1].

There has been a great interest in investigating the computational solutions of a unified STFDE of Fisher type (second order equations), which is obtained by replacing the second order diffusion term in by Riesz fractional derivatives and/or the first order term (the time derivative) by the generalized Riemann Liouville or Caputo fractional derivative, see [2–7]. In a

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recent previous work, analytic approximate solutions of a nonlinear STFDE of Fisher type equation is obtained (with Caputo, Riesz and generalized Riesz fractional derivatives) via applying optimal homotopy analysis method (OHAM) [8,9].

Numerous work have been done on fractional reaction–diffusion equations, some authors have discussed important nonlinear phenomena arise in physics, biology and chemistry, such as spatio-temporal pattern formation processes [10–13]. Great efforts have focused on the anomalous diffusion modeled by the fractional diffusion equation with spatial Riesz and Riesz–Feller fractional derivatives [1,14–17].

Also, higher order reaction–diffusion equations attracted the mathematicians and the physicists as well for their superiority in studying the phase transitions and pattern formation. There two famous models in the literature; the extended Fisher Kolmogorov (EFK) equation [18] and the Swift–Hohenberg (SH) equation [19]. The EFK equation and the SH equation are of fourth order type and have important applications in engineering and physics. Various analytical approximate methods are used to solve the SW equation, for example: the homotopy method [20] and the variational iteration method [21]. The EFK and SH equations involve the fourth-order spatial derivative, and if a higher order term added (sixth derivative), we obtain a new higher-order equation that reveals rich characteristics in the phase transition phenomenon and supports traveling wave solutions of a generalized phase-field model, see [22] and [23]. The sixth order equation studied by [23] is the monostable equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^6 u}{\partial x^6} + b\,\frac{\partial^4 u}{\partial x^4} + c\,\frac{\partial^2 u}{\partial x^2} + u - u^2, \qquad \gamma > 0 \tag{1}$$

where a, b and c are real constants .

In this work, the motivation is to give a recipe for obtaining the analytic approximate solution of the unified STFDE which give a generalization of the classical integer order phase field equation (1). We replace the space integer derivative by fractional Reisz derivative and the integer time derivative by fractional derivative in Caputo sense. Thus, the newly proposed unified STFDE is

$$\begin{cases} {}^{C}D_t^{\mu} u(x,t) = a\left(R_x^{\gamma}\right)^3 u + b\left(R_x^{\beta}\right)^2 u + c\,R_x^{\alpha} u + g(u), & -\infty < x < \infty,\ t > 0, \\ u(x,0) = f(x), \end{cases} \tag{2}$$

where R_x^{α}, R_x^{β}, and R_x^{γ} denotes the Riesz fractional derivative (in space) of order $\alpha,\ \beta$, and γ respectively. The fractional parameters α, β, and γ are restricted to the domain $(0, 1) \cup (1, 2)$. ${}^{C}D_t^{\mu}u(x,t)$ is the Caputo fractional derivative (in time) of order $\mu,\ 0 < \mu \le 1$. The functions g and f are continuous functions in u and x, respectively.

This article is organized as follows. In Section 2, basic definitions of fractional derivative operators involved are presented. In Section 3, we describe how the OHAM used to find the approximate homotopy series solution of the proposed problem (2). Two numerical examples are discussed in Section 4 and a conclusion of the work is included in Section 5.

2. Fractional derivatives and integrals

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0,\infty)$, and it is said to be in the space C_μ^m if $f^m \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f(t) \in C_\mu$, $\mu \ge -1$ is defined as

$$\begin{cases} J^{\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \ \alpha > 0,\ t > 0, \\ J^0 f(t) = f(t). \end{cases} \tag{3}$$

This integral operator has many useful properties, and we use the following one [24]

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \qquad \alpha > 0,\ \gamma > -1. \tag{4}$$

Definition 3. The fractional derivative in Riemann–Liouville sense of $f(t)$, m $\in \mathbb{N}$, $t > 0$ is defined as

$$\mathbf{D}_x^{\alpha} f(t) = \frac{d^m}{dx^m} J^{m-\beta} f(t), \qquad m-1 < \beta < m. \tag{5}$$

Definition 4. The fractional derivative in Caputo sense of $f(t) \in C_{-1}^m$, $m \in \mathbb{N}$, $t > 0$ is defined as

$${}_{C}D_t^{\mu} f(t) = \begin{cases} J^{m-\mu} \dfrac{d^m}{dt^m} f(t), & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{6}$$