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Transitions between superstatistical regimes: Validity, breakdown and applications

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HIGHLIGHTS

- We propose a simple model for a transition between two superstatistics in terms of infinitely divisible distributions.
- It is demonstrated that some financial time series can undergo transmutation of superstatistics.
- Numerical evidence for a transition between two superstatistics is provided by analyzing data for shareprice returns.
- Reasons for a breakdown of superstatistics in various systems are discussed.

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ABSTRACT

Superstatistics is a widely employed tool of non-equilibrium statistical physics which plays an important rôle in analysis of hierarchical complex dynamical systems. Yet, its "canonical" formulation in terms of a single nuisance parameter is often too restrictive when applied to complex empirical data. Here we show that a multi-scale generalization of the superstatistics paradigm is more versatile, allowing to address such pertinent issues as transmutation of statistics or inter-scale stochastic behavior. To put some flesh on the bare bones, we provide a numerical evidence for a transition between two superstatistics regimes, by analyzing high-frequency (minute-tick) data for share-price returns of seven selected companies. Salient issues, such as breakdown of superstatistics in fractional diffusion processes or connection with Brownian subordination are also briefly discussed.

1. Introduction

The concept of "emergence" plays an important role in modern statistical physics. One of the key characteristics of emergence is that the observed macroscopic-scale dynamics and related degrees of freedom differ drastically from the

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actual underlying microscopic-scale physics [1–3]. Such systems are often characterized by hierarchical structures of underlying dynamics. Superstatistics provides an explicit realization of this paradigm: It posits that the emergent behavior can be in many cases regarded as a superposition of several statistical systems that operate at different spatio-temporal scales [4]. The essential assumption of the superstatistics scenario is the existence of different spatio-temporal scales which are largely separated from each other, so that the system has enough time to relax to a local equilibrium state and to stay within it for some (phenomenologically relevant) time. The most common superstatistics applications are concerned with two characteristic scales only. In this framework a broad range of successful applications has been recently reported; these include hydrodynamic turbulence [5], non-stationary dynamical processes with time-varying multiplicative noise exponents [6], turbulence in quantum liquids [7], models of the metastatic cascade in cancerous systems [8], complex networks [9], ecosystems driven by hydro-climatic fluctuations [10], pattern-forming systems [11], wind velocity fluctuations [12,13], share price fluctuations [14,15] and high-energy physics [16,17].

In their recent paper [18], D. Xu and C. Beck have brought yet another twist into the superstatistics paradigm by suggesting that in certain financial time series one can observe a temporal breakdown from the log-normal-superstatistics (valid on the minute timescales) to the Gamma-superstatistics (valid on the daily timescales). This scale-dependent "transmutation" of statistics is a very interesting observation which is in many respects akin to a similar behavior known from the theory of continuous phase transitions. There the transmutation of statistics is imprinted in the behavior of the ensuing two-point auto-correlation function which in the disordered phase (above the critical temperature T_c) decays exponentially, while at the critical point it "transmutes" to a power-law decay. The latter signalizes the long-range correlated behavior that may lead to an infinite second (or even first) moment. In fact, the divergence of the lowest moments (as know, for instance, for certain Lévy distributions) implies the absence of underlying physical scales. In statistical physics, the absence of physical scales is interpreted as *scale invariance* which in turn invokes the notion of self-similarity which is a typical hallmark of the presence of a (stable) fixed point in the state space. The primary aim of this paper is to promote and elaborate the issue of "transmutation" of statistics in the superstatistics framework from the viewpoint of infinitely-divisible distributions and theory of critical phenomena. Our considerations will be bolstered with some explicit illustrations from financial market.

The structure of the paper is as follows. To set the stage we elucidate in the next section the inner workings of the superstatistics paradigm. We also outline potential generalizations of the "canonical" superstatistics scenario of Beck et al. [4,19] by considering more characteristic scales and different stable distributions for *prior*. In Section 3, we provide a numerical evidence for a transition between two superstatistics regimes, by analyzing high-frequency (minute-tick) data for share-price returns of seven selected companies. In doing so, we first employ the Multifractal Detrended Fluctuation Analysis (MFDFA) and Surrogate MFDFA to identify within each of the seven time series two (well separated) time scales with qualitatively different underlying dynamics. In the second step, we use the maximum likelihood method together with Kolmogorov–Smirnov, Cramér–von Mises and Anderson–Darling distribution distances to fit optimally the two scale statistical behaviors with existent universality classes in superstatistics. We show that four (out of seven chosen) share-price return time series can be quantitatively well understood as a transition between two superstatistics regimes. In particular, in all four cases we observed a transition from the log-normal-superstatistics (on ∼50 min scale) to Gamma-superstatistics (on ∼400 min scale). Some mechanisms of the formation of regime switching between different superstatistics are briefly discussed in Section 4. There we first comment on Xu–Beck's *synthetic model* and then show that an alternative explanation can be provided via multi-scale superstatistics with the help of the renormalization-group technique. In Section 5, we discuss some potential pitfalls that might happen in the data analysis when the superstatistics paradigm is used too naively or uncritically. Finally, Section 6 summarizes our results and discusses possible extensions of the present work. For the reader's convenience, we give in Appendix A a brief glossary of the companies whose share-price returns are considered in the text (see Table 3). The paper is also accompanied by the Supplementary material which collects additional supporting material not present but referred to in the main text.

2. Brief review of "canonical" superstatistics

In this section we briefly review some of the essentials from superstatistics that will be needed in the main body of the text. Following Refs. [4,19,20], we consider an intensive parameter $\beta \in [0, \infty)$ that appreciably changes over time scales that are much larger than the typical relaxation time of the local dynamics. The random variable β can be in practice identified, e.g., with the inverse temperature [4,19,20], energy dissipation rate (turbulent flow in Kolmogorov theory) [21], volatility (econophysics) [15], einbein (quantum relativistic particles) [16,17], etc. On an intuitive ground, one may understand the superstatistics by using the *adiabatic* Ansatz. Namely, the system under consideration, during its evolution, travels within its state space $\mathcal{M}$ (described by a state variable $A \in \mathcal{M}$) which is partitioned into small cells characterized by a sharp value of some intensive parameter β. Within each cell, the system is described by the conditional distribution $\wp(A|\beta)$. As β varies adiabatically from cell to cell according to some *mixing* (or *smearing*) distribution $f(\beta)$, the joint distribution of finding the system with a sharp value of β in the state A is $\wp(A, \beta) = \wp(A|\beta)f(\beta)$, which is nothing but the De Finetti–Kolmogorov relation. The resulting macro-scale (emergent) statistics $\wp(A)$ for finding system in the state A is obtain by eliminating the nuisance parameter β through marginalization, that is

$$\wp(A) = \int_0^\infty \mathrm{d}\beta \, \wp(A|\beta)f(\beta) \,. \tag{1}$$