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Ornstein–Uhlenbeck process with fluctuating damping

Chai Hok Eab [a], S.C. Lim [b,*]

[a] 386/23 Phetchbury 14, Bangkok 10400, Thailand
[b] Faculty of Engineering, Multimedia University, 63100 Cyberjaya, Selangor Darul Ehsan, Malaysia

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A B S T R A C T

This paper studies Langevin equation with random damping due to multiplicative noise and its solution. Two types of multiplicative noise, namely the dichotomous noise and fractional Gaussian noise are considered. Their solutions are obtained explicitly, with the expressions of the mean and covariance determined explicitly. Properties of the mean and covariance of the Ornstein–Uhlenbeck process with random damping, in particular the asymptotic behavior, are studied. The effect of the multiplicative noise on the stability property of the resulting processes is investigated.

1. Introduction

Studies of harmonic oscillator with randomly varying parameters dated back several decades ago. Disordered linear oscillator chain with random frequency or random mass was first considered by Dyson in 1953 [1]. Oscillator with random parameters has been studied in subsequent work [2–6]. Since then, a considerable amount of work covering oscillator with random frequency, random damping and random mass has appeared. A comprehensive review of the past work can be found in the book by Gitterman [7, and references therein].

Motivations for studying oscillators with fluctuating parameters come from their potential applications in modeling many natural and man-made systems. Brownian motion of a harmonic oscillator with random mass has been used to model systems where particles of the surrounding medium not only collide with Brownian particles, they can also adhere to them. Examples of such applications include diffusion of clusters with randomly growing masses [8,9], planet formation by dust aggregation [10–12], cluster dynamics during nucleation [13], growth of thin film [14], deposition of colloidal particles on an electrode [15], traffic flow [16,17], stock market prices [18,19], etc.

An oscillator in addition to the possibility of having random mass, it can also acquire fluctuating frequency. The influence of the fluctuating environment such as the presence of colored noise and viscosity can be reflected in fluctuating damping term or random oscillator frequency. Examples of applications of such processes include propagation and scattering of waves in a random medium [5,20] and turbulent ocean waves, low amplitude wind-driven waves on the ocean surface [21–23], financial markets in econophysics [24] and population dynamics in biology [25–27]. Recently, oscillator with fluctuating frequency has been used in the study of nano-mechanical resonators, where frequency fluctuations are resulted from molecules randomly adsorbing and desorbing onto or from the resonator, or diffusing along its surface [28–33].

* Corresponding author.
E-mail addresses: Chaihok.E@gmail.com (C.H. Eab), sclim47@gmail.com (S.C. Lim).

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The equations that describe one-dimensional Brownian motion of a free particle in a fluid are given by

$$\frac{dx(t)}{dt} = v(t), \tag{1a}$$

$$m\frac{dv(t)}{dt} + \gamma v(t) = \eta(t). \tag{1b}$$

Here m is the particle mass, and γ is the dissipative parameter of the viscous force of the fluid, and $\eta(t)$ is the random force due to the density fluctuation of the surrounding medium. (1a) and (1b) are known as the Langevin equations for the Brownian particle. The random force for the usual Langevin equation is given by the Gaussian white noise with zero mean and covariance

$$\langle \eta(t)\eta(s) \rangle = \delta(t-s). \tag{2}$$

Note that (1b) is also known as the Langevin equation for the Ornstein–Uhlenbeck velocity process $v(t)$.

For the Brownian motion of a damped harmonic oscillator driven by white noise, one has

$$m\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + \omega^2 x(t) = \eta(t), \tag{3}$$

where ω is the intrinsic oscillator frequency. If the viscous damping force $-\gamma v(t)$ is much larger than the inertial term md^2x/dt^2 in (3) such that to a good approximation the first term in (3) can be neglected. Such an overdamped limit results in a stochastic differential equation for the position $x(t)$:

$$\frac{dx(t)}{dt} = -\mu x(t) + \zeta(t), \tag{4}$$

where $\mu = \omega^2/\gamma$, and $\zeta(t) = \eta(t)/\gamma$. Eq. (4), which has the same form (except for the constant parameters) as the Langevin equation for the velocity of free Brownian motion (1b), can be regarded as the Langevin equation of a diffusion process in a harmonic oscillator potential $V(x) = \mu x^2/2$. The solution of (4) known as the Ornstein–Uhlenbeck position process is stationary in the large-time limit. This stationary Ornstein–Uhlenbeck process can be associated with the quantum mechanical oscillator. It is mainly for this reason that the stationary Ornstein–Uhlenbeck process is also known as oscillator process [34–37], especially in the books on path integral formulation of quantum theory [34–36].

The constant coefficient stochastic differential equation (3) for a damped harmonic oscillator driven by white noise can be generalized to

$$m(t)\frac{d^2x(t)}{dt^2} + \gamma(t)\frac{dx(t)}{dt} + \omega^2(t)x(t) = \eta(t), \tag{5}$$

where $m(t)$, $\gamma(t)$ and $\omega(t)$ can be deterministic or random functions of time. The case $m(t)$, $\gamma(t)$ and $\omega(t)$ are deterministic functions has been studied by many authors (see for examples [38–41, and references therein]). On the other hand, it is possible that the mass, damping coefficient and frequency can be fluctuating functions of time. The randomness may come from the random mass, fractal structure of the medium or random orientation of the Brownian particle, the external effect due to viscosity. The external randomness may cause by thermodynamic, electromagnetic and mechanical sources. This exterior fluctuation is crucial for random frequency. Gitterman [7] has provided a comprehensive discussion on oscillator with random mass, random damping and random frequency.

Harmonic oscillator with random damping and random frequency lately has attracted renewed interest due partly to its potential application to the modeling of certain nano-devices such as nanomechanical resonators and nanocantilever [29–33]. The case of harmonic oscillator with random damping [24,42–48] and random frequency [4,49–54] given by white noise and dichotomous Markov noise (the random telegraph process) is well-studied [4,42–45].

The main aim of this paper is to study Ornstein–Uhlenbeck process with random damping $\mu(t)$ based on a generalization of Eq. (4):

$$\frac{dx(t)}{dt} = -\mu(t)x + \zeta(t), \tag{6}$$

(6) can be regarded as the overdamped limit of a special case of (5) with $m(t)$ and $\gamma(t)$ equal to positive constants. Equivalently, it is the Langevin equation for the position $x(t)$ of the Brownian motion in a harmonic oscillator-like potential $V(x) = \mu(t)x^2/2$. We shall consider Eq. (6) with fluctuating damping due to two kinds of multiplicative noise, namely the dichotomous noise and fractional Gaussian noise. Note that random damping with fractional Gaussian noise has not been studied previously, though the case of dichotomous noise has been considered based on Eq. (3) instead of (6). Explicit solutions of the resulting stochastic processes will be obtained with the full expressions of the mean and covariance of the associated process calculated to facilitate the study of their asymptotic behavior. We also study the stability of these solutions.