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Complex dynamics of an SEIR epidemic model with saturated incidence rate and treatment

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HIGHLIGHTS

- A traditional SEIR model with saturated incidence rate and treatment is presented.
- The dynamics of the model is studied through stability and bifurcation point of view.
- Global stability of an endermic equilibrium is obtained by geometric approach.
- Optimal control problem is designed and numerical results are presented.

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ABSTRACT

In this paper, we describe the dynamics of an SEIR epidemic model with saturated incidence, treatment function, and optimal control. Rigorous mathematical results have been established for the model. The stability analysis of the model is investigated and found that the model is locally asymptotically stable when $\mathcal{R}_0 < 1$. The model is locally as well as globally asymptotically stable at endemic equilibrium when $\mathcal{R}_0 > 1$. The proposed model may possess a backward bifurcation. The optimal control problem is designed and obtained their necessary results. Numerical results have been presented for justification of theoretical results.

1. Introduction

The recent advancement in medical sciences enable us to control the spread of infections diseases in a better way than it was previously. Most of the infectious diseases associated with human population spread through direct contact or through some other sources, is still a threat to become epidemic in the world. The rapid developing and development in controlling tools for infectious diseases, as in time progresses, new harmful infectious diseases, with complicated nonlinear structure and nature, are entering into the population. The researchers and scientists not only from the field of medicine are always in search to find and develop some best tools and optimal way to control the spread of infectious diseases if there is no possibility of complete eradication.

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Mathematical models are often used to study the complicated nonlinear process and complex behavior such infectious diseases. An earlier work on the human epidemiology was done by Bernoulli [1] almost 250 years ago. Later on, a classical SIR epidemic model parented by Kermack and McKendric [2]. Recently, a lot of mathematical models have been presented on the control effect of infectious diseases ([3], Hosono and Ilyas [4], Hu et al. [5], Kar and Jana [6], Wang [7], Wang et al. [8]; Zhou and Fan [9], Khan et al. [10,11]), K. Hattaf et al. [12,13], M.C.Casaban et al. [14,15] and can be extended to fractional differential operators Atangana and Gomez [16,17]. The transmission of disease from an infected individuals to a healthy person is an important work to discuss and analyze. Usually this transmission is in the form of the mass action where the number of health individuals who contact the infected individuals goes to infected individuals class. First time, a saturated incidence has been introduced by Capasso and Serio [18]. In their work they considered the inhibition effect from the behavioral change of the healthy population or to consider the crowding effect of the infected individuals due to large amount of infected individuals presented in the system as in this type of disease transmission. Recently, using this type of incidence rate can be seen in [19], Sahu and Dhar [20], Kar and Jana [6] etc. Presently, different control strategies have been proposed in order to control the spread of infectious diseases. Regarding to this, vaccination and treatment controls are considered is a best tool. The work related to human epidemiology, some authors used vaccination control only, see for example [21–25], and some authors used the treatment control see for example [5,26,27]. Using both vaccination and treatment control we refer the reader to see the work in [6,28–31]. Present in this work, we formulate an epidemic model with treatment control only. The infectious diseases such as tuberculosis, measles, etc. are mostly controlled through some appropriate treatment control, see [7,32–34]. In general the consider treatment function is in linear form [31,6,30]. The work present in [7] is about the study of an epidemic model with limited resources for treatment. Recently, in [35] proposed a model with limited resources for vaccination control. A nonlinear treatment function has been considered by [3,19], we see the implementation of nonlinear treatment function.

Therefore, it is better to consider the treatment function in the form $T(u, I) = f(uI)$. Zhang and Liu [36] proposed a saturated type treatment function $T(I)$ of the form $T(I) = \gamma I/1 + \alpha I$, where $\gamma > 0$, $\alpha > 0$, to measure the effect of the infected being delayed for treatment. Similarly to their consideration, the treatment function consider by [37] is the treatment function as a function of both the control (u) and infectious human (I), and it is consider as $T(u, I) = (\phi uI)/1+\alpha uI$. It is found when either I or u is very low, the proposed treatment function approaches to a value near-zero and for a large value of I, it approaches to a finite limit. This type of treatment function would reflect the natural epidemic system and therefore, we intend to apply this treatment function in our proposed SEIR model. Here ϕ/α is the maximal supply of medical resource per unit time and $1/(1+\alpha uI)$ denotes the reverse effect of infected individuals which are delayed for treatment and it must have important effect on the spread of the disease [9]. An SIR model with saturated incidence and treatment function is considered by [37]. Inspire from the work of [37], we extended the SIR model into SEIR model. Following the procedure in [37] and obtain the dynamics of the SEIR epidemic model.

The organization of the paper follows: in Section 2, we formulate the proposed model in detail. In Section 3, we discuss the stability and bifurcation analysis of the proposed model. In the next Section 4, an optimal control problem is formulated and their analytical as well as numerical solutions is presented. Finally, in Section 5 we give the conclusion.

2. Model framework

This section describes the mathematical formulation of an SEIR epidemic model, denoting the total population size of an individuals by $N(t)$. The total population size of an individuals is subdivided into four different classes, namely, $S(t)$-susceptible, $E(t)$-exposed, $I(t)$-infected and $R(t)$ removed or recovered individuals. We consider the that the parasites of the diseases are transmitted to the susceptible populations by the direct contact with the infected populations. The modeling formulation is based on the following assumptions.

The population of susceptible individuals is increased at any time "t" by the recruitment rate Λ. We consider all the new births are susceptible and join the susceptible class $S(t)$. The population of susceptible individuals is decreased by the contact rate of susceptible with infected individuals by $\beta SI/1 + bI$, where $\beta \geq 0$ is the disease contact rate and b shows the saturation constant. The population of susceptible individuals is decreased by the natural death rate μ and increased by ψ (the recovered individuals susceptible once again). The governing equation for the above can be written as

$$\frac{dS}{dt} = \Lambda - \frac{\beta S(t)I(t)}{1 + bI(t)} - \mu S(t) + \psi R(t). \tag{1}$$

The population of exposed individuals is increased by the effective contact rate $\beta SI/1 + bI$ and decreased by the natural death rate μ and the transfer rate δ (the individuals in $E(t)$ get infection). The governing equation for this can be written as

$$\frac{dE}{dt} = \frac{\beta S(t)I(t)}{1 + bI(t)} - \mu E(t) - \delta E(t). \tag{2}$$

The population of infected individuals is increased by δ (individuals get infection). It is decreased by the saturated treatment function $\phi uI(t)/1 + \alpha uI(t)$ [37], where u represents the treatment control function, ϕ is positive quantity and α is a non-negative quantity. Moreover, the population of infected individuals is decreased by natural death rate μ, disease death rate γ and rate of recovery from infection θ. The time rate of change for this can be represented by the following equation

$$\frac{dI}{dt} = \delta E(t) - \frac{\phi uI(t)}{1 + \alpha uI(t)} - \mu I(t) - \gamma I(t) - \theta I(t). \tag{3}$$