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A stochastic chemostat model with an inhibitor and noise independent of population sizes☆

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HIGHLIGHTS

- A stochastic chemostat model with an inhibitor and noise is considered.
- The asymptotic behaviors of the solutions of the stochastic system are studied.
- The large noise can make the microorganisms become extinct almost surely.

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ABSTRACT

In this paper, a stochastic chemostat model with an inhibitor is considered, here the inhibitor is input from an external source and two organisms in chemostat compete for a nutrient. Firstly, we show that the system has a unique global positive solution. Secondly, by constructing some suitable Lyapunov functions, we investigate that the average in time of the second moment of the solutions of the stochastic model is bounded for a relatively small noise. That is, the asymptotic behaviors of the stochastic system around the equilibrium points of the deterministic system are studied. However, the sufficient large noise can make the microorganisms become extinct with probability one, although the solutions to the original deterministic model may be persistent. Finally, the obtained analytical results are illustrated by computer simulations.

1. Introduction

The chemostat has a significant role in mathematical biology and theoretical ecology, many authors have studied the chemostat models [1–7]. However, the inhibitor affects the nutrient uptake rate of one of the competitors in chemostat but is taken up by the other without ill effect. The Nalidixic acid used in the experiments of Hansen and Hubbell [8] is an inhibitor. Its effect on one strain of *E. coli* was essentially nil while the growth rate of the other was severely diminished [1]. Lenski

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and Hattingh have first proposed a chemostat model with an inhibitor as follows [9]:

$$\begin{cases} S'(t) = \left(S^0 - S(t)\right)D - \dfrac{m_1 S(t) x_1(t)}{a_1 + S(t)} f(p) - \dfrac{m_2 S(t) x_2(t)}{a_2 + S(t)}, \\ x_1'(t) = \left(\dfrac{m_1 S(t)}{a_1 + S(t)} f(p) - D\right) x_1(t), \\ x_2'(t) = \left(\dfrac{m_2 S(t)}{a_2 + S(t)} - D\right) x_2(t), \\ p'(t) = \left(p^0 - p(t)\right)D - \dfrac{\delta x_2(t) p(t)}{K + p(t)}, \end{cases} \tag{1.1}$$

where $S(t)$ represents the nutrient concentration at time t in the culture vessel, and $x_1(t), x_2(t)$, the concentration of two microorganisms, $p(t)$, the concentration of the inhibitor. All the parameters are positive constants. S^0 is the input concentration of the nutrient, p^0 is the input concentration of the inhibitor. D is the common washout rate. $\frac{m_i S(t)}{(a_i + S(t))}$ $(i = 1, 2)$ stand for the Monod growth functional response, where the terms m_i, a_i $(i = 1, 2)$ are the maximal growth rates of the competitors (without an inhibitor) and half-saturation constants, respectively. The dynamical behaviors of model (1.1) are completely determined by the break-even concentration λ_i $(i = 1, 2)$, where $\lambda_i = \frac{Da_i}{m_i - D}$. The δ and K have same effects for the inhibitor, with δ the maximal uptake rate by x_2 and K a half-saturation parameter. The function $f(p)$ stands for the degree of inhibitor of p on the growth rate of x_1. It adjusts the effective value of the parameter m_1; the quantity $m_1 f(p)$ represents the maximal growth rate of the microorganism x_1 if the concentration of the inhibitor is p. As noted in [1], the ability of x_2 to consume the inhibitor ($\delta > 0$) is of crucial importance. Lenski and Hattingh [9] refer to this ability of x_2 to “detoxify” the environment. Hsu and Waltman [10] reduced the problem (1.1) to a three-dimensional competitive system by using preliminary analysis. And by the theory of monotone flows, they obtained several global results. Global results fail when questions of multiple limit cycles cannot be answered. Especially, they proposed that the chemostat with inhibitor can model competition between two populations of microorganisms, where one strain is resistant to an antibiotic or competition in detoxification, a system where one strain can take up the pollutant while the other is inhibited by it.

Assume that the function $f(p)$ in (1.1) satisfies:
(i) $f(p) \geq 0, f(0) = 1$;
(ii) $f'(p) < 0, p > 0$.

Denote

$$\lambda_0 = \frac{Da_1}{m_1 f(p^0) - D}.$$

Referring to [1] or [10], we give the following Lemmas about the system (1.1).

Lemma 1.1. *If $m_i \leq D$ or $m_i > D$, $\lambda_i > S^0 (i = 1, 2)$, the extinct equilibrium $E_0 = (S^0, 0, 0, p^0)$ exists, and the E_0 is asymptotically stable with $\lambda_0 > S^0$.*

Lemma 1.2. *If $m_i > D$, $0 < \lambda_2 < \lambda_1 < S^0$, the rest point $E_1 = (S_1, 0, \hat{x}_2, \hat{p})$ exists with $S_1 = \lambda_2$, $\hat{x}_2 = S^0 - \lambda_2$ and $\hat{p}$ the positive root of $D(p^0 - p)(K + p) - \delta(1 - \lambda_2)p = 0$. The E_1 is asymptotically stable if $\lambda_2 < \frac{Da_1}{m_1 f(\hat{p}) - D}$.*

Lemma 1.3. *If $m_i > D$, $0 < \lambda_1 < \lambda_2 < S^0$ and $0 < \lambda_0 < \lambda_2$, the system (1.2) has a asymptotically stable boundary equilibrium $E_2 = (\lambda_0, S^0 - \lambda_0, 0, p^0)$.*

Lemma 1.4. *The positive equilibrium $E^* = (S^*, x_1^*, x_2^*, p^*)$ exists if $m_i > D$, $\lambda_1 < \lambda_2 < S^0$ and $\lambda_2 < \lambda_0$. According to the Routh–Hurwitz , the E^* will be asymptotically stable if the following conditions holds:*

$$\text{(i)} \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0. \quad \text{(ii)} \begin{vmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0. \quad \text{(iii)} \begin{vmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{vmatrix} > 0$$

where

$$a_1 = 2D + \frac{K\delta x_2^*}{(K + p^*)^2}, \quad a_2 = D\left(\frac{a_2 m_2 x_2^*}{(a_2 + S^*)^2} + \frac{a_1 m_1 x_1^* f(p^*)}{(a_1 + s^*)^2} + D + \frac{K\delta x_2^*}{(K + p^*)^2}\right),$$

$$a_3 = D\left(D + \frac{K\delta x_2^*}{(K + p^*)^2}\right)\left(\frac{a_2 m_2 x_2^*}{(a_2 + S^*)^2} + \frac{a_1 m_1 x_1^* f(p^*)}{(a_1 + S^*)^2}\right), \quad a_4 = D \cdot \frac{a_2 m_2 x_2^*}{(a_2 + S^*)^2} \cdot \frac{-\delta p^*}{K + p^*} \cdot \frac{m_1 S^* x_1^*}{a_1 + S^*} \cdot f'(p^*).$$

As mentioned above, we notice that the chemostat models are described by the deterministic models. This is valid only at the macroscopic scale, that is, the stochastic effects can be neglected or averaged out, in view of the law of large number. However, the natural growth of species are inevitably affected by random environment noise. It turns out that a reasonable