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Clusters' size-degree distribution for bond percolation

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HIGHLIGHTS

- The generating function of the clusters' degrees distribution for bond percolation as $q \to 1$ limit of modified Potts model is found.
- For bond percolation on Bethe lattice and complete graph we found the clusters' size-degree distributions.
- The analytical representation would simplify the numerical calculations of the clusters' degrees distributions.

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ABSTRACT

To address some physical properties of percolating systems it can be useful to know the degree distributions in finite clusters along with their size distribution. Here we show that to achieve this aim for classical bond percolation one can use the $q \to 1$ limit of suitably modified q-state Potts model. We consider a version of such model with the additional complex variables and show that its partition function gives the bond percolation's generating function for the size and degree distribution in the $q \to 1$ limit. For the first time we derive this distribution analytically for bond percolation on Bethe lattices and complete graph. The possibility to expand the applications of present method to other clusters' characteristics and to models of correlated percolation is discussed.

1. Introduction

Modern percolation theory knows a lot about the structure and geometry of percolation clusters on a number of graphs [1–3]. It is also known how its structure changes with the growth of occupation probability — it acquires high-density [4,5], and bootstrap (k-core) backbones [6,7] having different sites' degrees (the numbers of bonds attached to a site). In what concerns the finite clusters in percolation, Monte Carlo studies of its average radius, perimeter, fractal dimension, average shape and density profile have been fulfilled for some lattices [8–12]. Yet most important and most studied is the clusters' size distribution ν_s as finding it we get all set of critical indices for percolation transition as well as its order parameter. In its turn, the size distribution ν_s is defined by the graph's numbers (per site) $\nu_{s,t}$ of distinct clusters (lattice animals) with a given size s and properly defined perimeter t [3,10,13]. Thus $\nu_{s,t}$ (or ν_s) for a graph is all that needed for standard description of the classical percolation transition on it. So many existing studies of finite clusters in percolation are devoted to numerical determination of $\nu_{s,t}$ and ν_s on various lattices, see, for example, [10,13–16]. Also ν_s is obtained analytically for Bethe lattice [17] and complete graph [18].

The relation of the $q \to 1$ limit of q-state Potts model to the ν_s generating function [18] is of great help in these studies. Later the modified Potts models were introduced which are related to the generating functions of $\nu_{s,b}$ [19] and $\nu_{s,b,t}$ [20] b being the number of bonds in a cluster. These works demonstrated for the first time that detailed structural characteristics of finite clusters could be found with the methods of usual statistical mechanics.

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To expand this approach one may think of obtaining distributions of other structural characteristics of s-site clusters in the Potts model framework. In particular, there are many properties of random systems, which crucially depend on the compactness of finite clusters. They are the ability to participate in chemical reactions, resilience under random removal of their bonds [21] and possibility to acquire net magnetic moment [4,5], to name a few. To address such problems it is useful to know the degree distributions in finite clusters along with their size distribution. Here we show that to achieve this aim for the classical bond percolation one can use the $q \to 1$ limit of suitably modified q-state Potts model. We consider a version of such model with the additional complex variables and show that its partition function gives the bond percolation's generating function for size and degree distribution in the $q \to 1$ limit (Section 2). For the first time we derive this distribution analytically for bond percolation on Bethe lattices (Section 3) and complete graph (Section 4) thus providing new exact results for these textbook models. In Section 5 we discuss the possibility to expand the applications of present method to other clusters' characteristics and to models of correlated percolation.

2. Modified Potts model

Let us first recall the established procedure to find generating function for clusters' size distribution in bond percolation with the $q \to 1$ limit of q-state Potts model [18]. For some graph with N sites and E edges on which bonds are placed with probability p consider the Hamiltonian

$$H_0(h, \boldsymbol{\sigma}) = \sum_{\langle i,j\rangle} \ln(1-p)\left(\delta_{\sigma_i,\sigma_j} - 1\right) + \sum_{i=1}^{N} h\left(\delta_{\sigma_i,1} - 1\right)$$

where paired Potts interactions are assigned to each edge of the graph. The partition function of this Potts model is

$$Z_0(q,h) = Tre^{-H_0(h,\boldsymbol{\sigma})} = Tr\prod_{\langle i,j\rangle}\left(1-p+p\delta_{\sigma_i,\sigma_j}\right)\prod_i e^{h\left(1-\delta_{\sigma_i,1}\right)}$$

where Tr denotes the sum over all spins $\sigma_i = \{1,\ldots,q\}$. It can be represented as sum over all bond configurations on the graph

$$Z_0(q,h) = \sum_C p^B(1-p)^{E-B}\prod_{clusters}\sum_{\sigma=1}^{q} e^{s_{cl}\left(1-\delta_{\sigma,1}\right)} = \sum_C p^B(1-p)^{E-B}\prod_s\left[1+(q-1)e^{sh}\right]^{N_s}.$$

Here B is a number of bonds in a given configurations, s_{cl} is a cluster's size and N_s is the number of s-site clusters in a configuration. Hence,

$$G_0(h) = \lim_{q\to1} N^{-1}\frac{d\ln Z_0}{dq} = \sum_s \nu_s e^{hs}, \quad \nu_s = \left\langle\frac{N_s}{N}\right\rangle_C.$$

Here $\langle\cdots\rangle_C$ means the average over the random configurations of bonds, which occupy the graph's edges with probability p, so $G_0(h)$ is the generating function for the average clusters' size distribution in classical bond percolation.

This derivation relies heavily on the useful property of the model to make equal all the spins in the cluster. Here we intend to use it to obtain the generation function for the degree distribution

$$\nu_{s_0,\mathbf{s}} = \lim_{N\to\infty}\left\langle\frac{N_{s_0,\mathbf{s}}}{N}\right\rangle_C \tag{1}$$

where s_0 and $\mathbf{s} = \{s_1,\ldots,s_z\}$ with $s_k$defining the number of sites of degree k (i.e., with k bonds attached) in a cluster, z is the maximal number of edges attached to the site in the graph and $N_{s_0,\mathbf{s}}$ is the number of clusters with the given degree distribution $\{s_0,\mathbf{s}\}$ in a given bond configuration. We separate s_0 for future convenience as it refers to the special clusters composed of single isolated sites. Apparently, $N_{s_0,\mathbf{s}}$ have the form

$$N_{s_0,\mathbf{s}} = N_1\delta(s_0,1) + N_{\mathbf{s}}\delta(s_0,0)$$

where N_1 is the number of single-site clusters and $N_{\mathbf{s}}$ is that of clusters with more than one site which do not have sites with zero degree.

Accordingly, the generation function depend on $(z+1)$-component vector $\{h_0,\mathbf{h}\}$ as follows

$$G(h_0,\mathbf{h}) = \sum_{s_0,\mathbf{s}}\nu_{s_0,\mathbf{s}}e^{s_0h_0+\mathbf{sh}} = \nu_1e^{h_0} + \sum_{\mathbf{s}}\nu_{\mathbf{s}}e^{\mathbf{sh}} = \nu_1e^{h_0} + \tilde{G}(\mathbf{h}) \tag{2}$$

$$\nu_1 = \lim_{N\to\infty}\left\langle\frac{N_1}{N}\right\rangle_C, \quad \nu_{\mathbf{s}} = \lim_{N\to\infty}\left\langle\frac{N_{\mathbf{s}}}{N}\right\rangle_C.$$

On the regular graphs with single coordination number z for all sites

$$\nu_1 = (1-p)^z$$