Physica A 491 (2018) 1001–1013

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Conformable derivative approach to anomalous diffusion

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HIGHLIGHTS

- This work made the first attempt to describe the anomalous diffusion based on the conformable derivative.
- Analytical solutions of the conformable derivative model are obtained in terms of Error function and Gauss kernel.
- The conformable derivative model agrees better with experimental data than the conventional diffusion equation.
- The conformable derivative model can effectively predict the sub-diffusion process.

ARTICLE INFO

Article history:
Received 6 June 2017
Received in revised form 1 September 2017

Keywords:
Anomalous diffusion
Conformable derivative
Gauss kernel
Mean square displacement
Error function

ABSTRACT

By using a new derivative with fractional order, referred to conformable derivative, an alternative representation of the diffusion equation is proposed to improve the modeling of anomalous diffusion. The analytical solutions of the conformable derivative model in terms of Gauss kernel and Error function are presented. The power law of the mean square displacement for the conformable diffusion model is studied invoking the time-dependent Gauss kernel. The parameters related to the conformable derivative model are determined by Levenberg–Marquardt method on the basis of the experimental data of chloride ions transportation in reinforced concrete. The data fitting results showed that the conformable derivative model agrees better with the experimental data than the normal diffusion equation. Furthermore, the potential application of the proposed conformable derivative model of water flow in low-permeability media is discussed.

1. Introduction

Anomalous diffusion is a complex transport problem in many physical, chemical and biological phenomena, for instance, macromolecules transport in biological cells, water seepage and contaminants migration in porous media. A growing efforts has been devoted to modeling of the anomalous diffusion within the literatures [1,2]. The normal diffusion equation, which is usually characterized by the Fick's law, is not adequate for the anomalous diffusion [3–5]. In contrast to the normal diffusion, anomalous diffusion is not a diffusion process complying with the Gaussian statistics any more. It is well known that the mean square displacement (MSD) of normal diffusion is a linear function of time, however, the anomalous diffusion follows a non-linear function of time, *e.g.*, a power law of time with the exponents higher or lower than 1.0. Additionally, theoretical analysis and experimental investigation have been documented that the anomalous diffusion exhibits the features of time

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https://doi.org/10.1016/j.physa.2017.09.101
0378-4371/

dependence and heavy-tails [6]. Consequently, many researchers devoted to depicting such anomalous diffusion employing the fractional derivative [7–10].

The fractional derivative generalized the classical derivative of integer order to a differential operator of arbitrary order. Due to its non-local behavior which yields the memory effects and history dependence, in the past decades, the fractional derivative has gained remarkable applications in various areas of science and engineering [11,12] such as the time-dependent characteristics of rocks [13,14] and composite materials [15], fluid mechanics [16], nanotechnology [17], control system [18], electrical circuits [19], thermo-elasticity [20], nuclear reactor dynamics [21] and biology [22].

Moreover, the fractional derivative approach to anomalous diffusion can be found in many literatures. [23] considered fractional diffusion in inhomogeneous media. [24] proposed a variable order time-fractional derivative model to describe the chloride diffusion in concrete. [25] established a time-fractional derivative diffusion model for chloride ions subdiffusion. More information about the fractional diffusion equations, we refer to the monograph introduced by Povstenko [26]. It should be emphasized that the analytical solutions of the fractional diffusion equations belonging to each of the above mentioned literatures are often not available, and should deduce approximate solutions from the complicated numerical methods. These difficulties are due to the inconsistencies in comparing fractional derivative definitions with integer derivative as a local operator. Non-local fractional derivative operators do not obey some basic properties of integer derivative such as product rule, quotient rule, chain rule and semigroup property. The lack of these basic properties come with inconvenience in mathematical handle. For the sake of conquering this defect, we make utmost efforts to develop a simple and effective model to depict the anomalous diffusion.

This work makes an attempt to describe the subdiffusion based on the conformable derivative, which is a newly definition of derivative with fractional order. The analytical solutions of the conformable derivative model in terms of Gauss kernel and Error function are presented. The power law of the mean square displacement (MSD) for the conformable diffusion model is studied invoking the time-dependent Gauss kernel. The parameters related to the conformable derivative model are determined by the experimental data of chloride ions transportation in reinforced concrete. The fitting results indicated that the conformable derivative model is in better agreement with the experimental data than the conventional diffusion equation. Sensitivity studies were also carried out, illustrating the effects of fractional derivative order on the chloride penetration. Furthermore, the potential application of the proposed conformable derivative model in water flow in low-permeability media is discussed.

2. Conformable derivative model

2.1. Definition and properties of conformable derivative

The most popular definitions of fractional derivatives among them are Riemann–Liouville and Caputo. Although they are linear operators and possess some fine properties, they do not inherit all the operational behaviors from the typical first derivative, such as product rule, quotient rule, chain rule and semigroup property. These inconsistencies lead to the development of the local fractional derivative whose most properties coincide with classical integer derivative [27–30]. Khalil et al. [29] introduced a new local fractional derivative which is well-behaved and complies with the computational relationships of the first derivative, called conformable derivative. Because of its effectiveness and applicability, conformable derivative has been used in various field such as Newton mechanics [31], quantum mechanics [32], arbitrary time scale problems [33], diffusive transport [34,35] and stochastic process [36]. Consequently, the interest in the conformable derivative has been increasing and it is worthwhile to explore in this new research area. The conformable derivative of a function $f(t):[0,\infty)\rightarrow\mathbb{R}$ for all $t>0$ with order $\alpha\in(0,1]$ is defined by

$$T_\alpha f(t)=\lim_{\varepsilon\rightarrow 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{1}$$

and the conformable derivative at 0 is given by $(T_\alpha f)(0)=\lim_{t\rightarrow 0}(T_\alpha f)(t)$.

Furthermore, Khalil et al. [29] stated that the relationship between the conformable derivative and the first derivative can be represented as

$$T_\alpha f(t)=t^{1-\alpha}\frac{df(t)}{dt}. \tag{2}$$

It is obvious that the conformable derivative is a local operator and coincides with the classical first derivative when the differential order is $\alpha=1$. Recently, a physical interpretation of the conformable derivative is given by Zhao and Luo [37]. They generalized the definition of the conformable derivative to general conformable derivative by means of linear extended Gâteaux derivative, and employed this definition to indicate that the physical interpretation of the conformable derivative is a modification of classical derivative in direction and magnitude. More basic properties and main results on conformable derivative, we refer to [29,38].