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Analytical and numerical treatment of the heat conduction equation obtained via time-fractional distributed-order heat conduction law

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HIGHLIGHTS

- Classical heat conduction equation is generalized by considering fractional distributed-order Cattaneo type heat conduction law.
- The Cauchy problem for system of energy balance equation and constitutive heat conduction law is treated analytically and numerically.
- Good agreement of both methods in cases of multi-term and power-type distributed-order heat conduction laws is found.
- Temperature and heat flux spatial profiles correspond to the propagation of heat waves.

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ABSTRACT

Generalization of the heat conduction equation is obtained by considering the system of equations consisting of the energy balance equation and fractional-order constitutive heat conduction law, assumed in the form of the distributed-order Cattaneo type. The Cauchy problem for system of energy balance equation and constitutive heat conduction law is treated analytically through Fourier and Laplace integral transform methods, as well as numerically by the method of finite differences through Adams–Bashforth and Grünwald–Letnikov schemes for approximation derivatives in temporal domain and leap frog scheme for spatial derivatives. Numerical examples, showing time evolution of temperature and heat flux spatial profiles, demonstrate applicability and good agreement of both methods in cases of multi-term and power-type distributed-order heat conduction laws.

1. Introduction

Heat conduction in one-dimensional rigid material is considered on infinite spatial domain $x \in \mathbb{R}$ and for time $t > 0$. Generalization of the heat conduction equation is considered by treating two different processes in a material. The first one is heating, described by the energy balance equation

$$\rho c \frac{\partial}{\partial t} T(x,t) = -\frac{\partial}{\partial x} q(x,t), \tag{1}$$

where ρ is used to denote the material density, c is the specific heat capacity, while T and q denote temperature and heat flux respectively. The second one is heat conduction, described by the Cattaneo type time-fractional distributed-order heat

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conduction law

$$\int_0^1 \phi(\gamma)\, {}_0^c\mathrm{D}_t^\gamma q(x,t)\,\mathrm{d}\gamma = -\lambda \frac{\partial}{\partial x} T(x,t), \tag{2}$$

where ϕ is the constitutive function or distribution, λ is the thermal conductivity and ${}_0^c\mathrm{D}_t^\gamma$ is the operator of Caputo fractional differentiation of order $\gamma \in (0,1)$, defined by

$${}_0^c\mathrm{D}_t^\gamma y(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} *_t \frac{\mathrm{d}y(t)}{\mathrm{d}t},$$

see [1], with $*_t$ denoting the convolution in time: $f(t) *_t g(t) = \int_0^t f(u)\, g(t-u)\,\mathrm{d}u$.

Rather than obtaining and solving a single heat conduction equation, the aim is to solve the system of equations consisting of energy balance equation (1) and constitutive equation (2), subject to initial

$$T(x,0) = T_0(x), \quad q(x,0) = 0, \quad x \in \mathbb{R}, \tag{3}$$

and boundary conditions

$$\lim_{x\to\pm\infty} T(x,t) = 0, \quad \lim_{x\to\pm\infty} q(x,t) = 0, \quad t > 0. \tag{4}$$

In particular, two special cases of the heat conduction law (2) will be examined: multi-term heat conduction law, obtained for the choice of constitutive distribution as

$$\phi(\gamma) = \tau_0\,\delta(\gamma-\alpha_0) + \sum_{\nu=1}^{N} \tau_\nu\,\delta(\gamma-\alpha_\nu), \quad \begin{array}{c} 0 \le \alpha_0 < \cdots < \alpha_N < 1, \\ \tau_0, \tau_1, \ldots, \tau_N > 0, \end{array} \tag{5}$$

consisting of at least two terms, where δ is used to denote the Dirac δ-distribution, and power-type distributed-order heat conduction law, obtained for the choice of constitutive function as

$$\phi(\gamma) = \tau^\gamma, \quad \tau > 0. \tag{6}$$

The constitutive equation (2) represents the generalization of known heat conduction laws such as Fourier, Cattaneo, fractional Cattaneo, which are obtained by choosing the constitutive distribution as

$$\phi(\gamma) = \delta(\gamma), \quad \phi(\gamma) = \tau\delta(\gamma-1) + \delta(\gamma), \quad \phi(\gamma) = \tau\delta(\gamma-\alpha) + \delta(\gamma), \tag{7}$$

respectively.

Underlying the physical motivation for considering system of the energy balance equation (1) and the constitutive Cattaneo type time-fractional distributed-order heat conduction law (2) as the model for heat conduction phenomena in rigid conductor, it should be noted that the energy balance equation reflects the physical process of material heating due to the initial spatial temperature distribution inducing, due to existence of the temperature gradient, the occurrence of heat flux. Being valid for any rigid heat conducting material, this equation is not generalized. Material properties are described by the constitutive heat conduction law. In the case of classical heat conduction equation, the Fourier heat conduction law accounts for local, both in space and time, occurrence of heat flow induced by the temperature gradient. In the case of materials displaying memory effects, heat flux at the present time-instant is influenced by the whole history of the temperature gradient and can be represented by the integral over time of the heat flux multiplied by some material dependent kernel. In the case of classical and fractional Cattaneo laws the kernels are respectively of exponential and Mittag-Leffler type, see [2]. In the present work, heat conduction law is further generalized in order to model wider range of history-dependent materials. Further, system of Eqs. (1) and (2) can be reinterpreted in order to model mass transport phenomena as well, where the first equation represents the mass (more precisely number of particles) balance law and the second one generalizes the first Fick law for materials displaying memory effects, where diffusion flux depends on the history of concentration gradient.

The approach of considering generalized heat conduction equation through system of balance and constitutive equation is also adopted in [3] within the classical theory using the analogy with circuits and extending the results within the theory of fractional calculus in [4]. Anomalous transport processes through space and time fractional generalizations of the Cattaneo heat conduction law are studied in [5]. Time and space fractional heat conduction of Cattaneo type is studied, analytically on infinite domain in [2] and with physical justification for non-locality introduction in [6–8]. Heat conduction problem with the Riesz space fractional generalization of the Cattaneo–Christov heat conduction model is numerically treated in [9]. Heat conduction problems with different heat conduction laws in terms of the classical theory are reviewed in [10], while in [11] there is a collection of heat conduction problems within the theory of fractional calculus.

By combining the energy balance equation (1) with the constitutive equation (2), where constitutive distributions are given by (7), the classical heat conduction, telegraph and fractional telegraph equations are obtained as

$$\frac{\partial T}{\partial t} = \mathcal{D}\frac{\partial^2 T}{\partial x^2}, \quad \tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathcal{D}\frac{\partial^2 T}{\partial x^2} \quad \text{and} \quad \tau\, {}_0^c\mathrm{D}_t^{\alpha+1} T + \frac{\partial T}{\partial t} = \mathcal{D}\frac{\partial^2 T}{\partial x^2}, \tag{8}$$