Physica A 491 (2018) 810–819

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Analytical solutions of the space–time fractional Telegraph and advection–diffusion equations

Ashraf M. Tawfik [a,b,*], Horst Fichtner [a], Reinhard Schlickeiser [a], A. Elhanbaly [b]

[a] Institut für Theoretische Physik IV, Ruhr-Universität Bochum, Universitätsstrasse 150, D-44780 Bochum, Germany
[b] Theoretical Physics Research Group, Mansoura University, Mansoura 35516, Egypt

ARTICLE INFO

Article history:
Received 19 July 2017
Received in revised form 21 September 2017
Available online 7 October 2017

Keywords:
Fractional calculus—diffusion-energetic particles

ABSTRACT

The aim of this paper is to develop a fractional derivative model of energetic particle transport for both uniform and non-uniform large-scale magnetic field by studying the fractional Telegraph equation and the fractional advection–diffusion equation. Analytical solutions of the space–time fractional Telegraph equation and space–time fractional advection–diffusion equation are obtained by use of the Caputo fractional derivative and the Laplace–Fourier technique. The solutions are given in terms of Fox's H function. As an illustration they are applied to the case of solar energetic particles.

1. Introduction

An increasing interest for the analysis and application of fractional differential equations has developed in recent years, because such equations describe anomalous transport processes and have applications in different topics of science [1–5]. Particularly, in astrophysics fractional differential equations are used to describe the motion of charged particles, such as cosmic rays [6] and solar energetic particles [7–9], in turbulent electromagnetic fields. The solution of the probability distribution function at a certain time can be successfully obtained by using fractional calculus with the theory of derivatives and integrals of fractional order. If α and β are the fractional orders for time and space, then the space–time fractional derivatives are related to the anomalous waiting time distribution and the power-law step length distribution [10].

Anomalous diffusion is characterized by a diffusion coefficient and the mean square displacement of a diffusing species is $\langle z^2(t)\rangle \sim t^\alpha$. For so-called Lévy flights with stable index $\alpha > 1$ (superdiffusion) the mean square displacement is diverging, see, e.g. [11]. In case of a slow process $0 < \alpha < 1$ it is called subdiffusive. Subdiffusion has been observed in many real physical systems such as highly ramified media in porous systems, spatially disordered or fractal media and their temporal fluctuations [12–14]. It is well-known that subdiffusion equations in terms of fractional derivatives can be obtained from Continuous Time Random Walk models [15] with long-tailed waiting time distributions and in general the anomalous regimes are characterized by non-Gaussian statistics like Lévy statistics, which encompasses probability distributions with power-law tails. For a recent review on anomalous diffusion see [16], where the differences to normal diffusion are discussed in detail, with an emphasis on the non-ergodicity. The latter is introduced by fractional time and space operators and refers to the difference of the long time and ensemble averages of physical observables, see also Ref. [17].

The normal diffusive evolution of a particle distribution is governed by the Fokker–Planck equation [18]. The Fokker–Planck equation describes the transport due to numerous process. Approximations are usually required to solve the Fokker–Planck equation [19]. In that paper the diffusion approximations was discussed along with the so-called modified Telegraph equation, which includes the standard Telegraph and advection–diffusion equations. The applications of Telegraph

* Corresponding author at: Institut für Theoretische Physik IV, Ruhr-Universität Bochum, Universitätsstrasse 150, D-44780 Bochum, Germany.
E-mail address: Ashraf.Attia@ruhr-uni-bochum.de (A.M. Tawfik).

https://doi.org/10.1016/j.physa.2017.09.105
0378-4371/

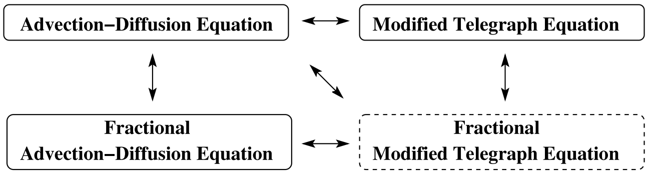

Fig. 1. The relations between the advection–diffusion equation, the modified Telegraph equation, and their fractional counter parts. The dashed line indicates the extension being discussed in this paper.

equations have a wide range in astrophysical and plasma physics [20]. They were extended to be applicable in irreversible thermodynamics [21], cosmological models [22] and shock waves in rigid heat conductors [23]. The Telegraph equation is also known to approximate the Fokker–Planck equation when the pitch-angle scattering of energetic particles is strong enough. By neglect of the relaxation time the modified Telegraph equation reduces to the advection–diffusion equation which also has various applications in astrophysics and plasma transport theory [19].

The advection–diffusion equation and the modified Telegraph equation are important tools for the description of the evolution of energetic particle distributions [24,25] but they do not comprise the treatment of anomalous transport. As stated above, they can, however, be generalized to be applicable in such cases within the framework of fractional calculus, i.e. by replacing the ordinary derivatives with fractional derivatives [26–28]. While this has been done in [7] for the fractional advection–diffusion equation in terms of the Riesz derivative, here we formulate and solve both the fractional advection–diffusion equation and the fractional modified Telegraph equation in terms of the Caputo derivative, which is to be used for bounded spatial domains [29]. Since the fractional modified Telegraph equation contains the other equations as special cases, as visualized in Fig. 1, we provide a flexible and comprehensive new modeling tool.

The paper is organized as follows. In Section 2, the fractional Telegraph and advection–diffusion equations are obtained from the Fokker–Planck equation with considering the space–time fractional form of two equations, and by using the Laplace–Fourier transform technique we obtain the analytical solutions of the space–time fractional Telegraph and space–time fractional advection–diffusion equations in terms of the Fox H function. In Section 3, as an illustration we study the fractional-order effect on the behavior of some astrophysical phenomena. Finally, all results are summarized is Section 4.

2. The Fokker–Planck equation and the diffusion approximations

The Fokker–Planck equation for the energetic particle distribution function including the effects of so-called pitch-angle scattering and adiabatic focusing is given by [20]

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial z} + \frac{v}{2L}(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu}\right). \tag{1}$$

Here $f(z, \mu, t)$ is the phase-space distribution function of energetic particles, t is time, $-1 \le \mu \le 1$ is the cosine of the particle pitch angle, v is the particle speed, z is the distance along the mean magnetic field (B_0), $L = -B_0/(\partial B_0/\partial z)$ is the adiabatic focusing length and $D_{\mu\mu}$ is the Fokker–Planck coefficient for pitch-angle scattering. The above Fokker–Planck equation is often referred to as the Klein–Kramers equation.

We divide the distribution function into an isotropic density F and an anisotropic component g: $f(z, \mu, t) = F(z, t) + g(z, \mu, t)$ where $\int_{-1}^{1} g \, d\mu = 0$.

Assuming that $|g| \ll F$, an approximate expression for g can be found and substituted into Eq. (1), integrated with respect to μ. Depending on the accuracy of the expression for g the result is the approximated Telegraph equation for focused transport [30]. Here we introduce the modified Telegraph equation for isotropic density:

$$\frac{\partial}{\partial t}F(z,t) + \tau\frac{\partial^2}{\partial t^2}F(z,t) = \kappa_\parallel \frac{\partial^2}{\partial z^2}F(z,t) + \xi\kappa_\parallel \frac{\partial}{\partial z}F(z,t), \tag{2}$$

where $\xi\kappa_\parallel$ is the coherent speed [31], $\kappa_\parallel$ is the parallel diffusion coefficient, $\xi = 1/L$ is the focusing strength and τ is the relaxation time.

In the absence of the focusing effect ($\xi = 0$), the last term in Eq. (2) vanishes. In this case the modified Telegraph equation will be reduced to the Telegraph diffusion equation and has the form

$$\frac{\partial}{\partial t}F(z,t) + \tau\frac{\partial^2}{\partial t^2}F(z,t) = \kappa_\parallel \frac{\partial^2}{\partial z^2}F(z,t). \tag{3}$$

In an alternative approximation we neglect the relaxation time in Eq. (2) by taking $\tau = 0$ and the modified Telegraph reduces to the advection–diffusion equation of the form

$$\frac{\partial}{\partial t}F(z,t) = \kappa_\parallel \frac{\partial^2}{\partial z^2}F(z,t) + B\,\frac{\partial}{\partial z}F(z,t), \tag{4}$$

where $\kappa_\parallel$ is a diffusion coefficient and $B = \xi\kappa_\parallel$ is a constant advection speed.