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Space–time fractional diffusion equation using a derivative with nonsingular and regular kernel

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HIGHLIGHTS

- The Atangana–Baleanu fractional derivatives in Caputo and Riemann–Liouville sense are applied to develop a new representation of the fractional diffusion equations.
- The generalization of the equations in space–time exhibits anomalous behavior.
- To keep the dimensionality an auxiliary parameter is introduced.
- Based on the Mittag-Leffler function new behaviors for concentration were obtained.

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ABSTRACT

In this paper, using the fractional operators with Mittag-Leffler kernel in Caputo and Riemann–Liouville sense the space–time fractional diffusion equation is modified, the fractional equation will be examined separately; with fractional spatial derivative and fractional temporal derivative. For the study cases, the order considered is $0 < \beta, \gamma \leq 1$ respectively. In this alternative representation we introduce the appropriate fractional dimensional parameters which characterize consistently the existence of the fractional space–time derivatives into the fractional diffusion equation, these parameters related to equation results in a fractal space–time geometry provide a new family of solutions for the diffusive processes. The proposed mathematical representation can be useful to understand electrochemical phenomena, propagation of energy in dissipative systems, viscoelastic materials, material heterogeneities and media with different scales.

1. Introduction

Fractional calculus (FC) has attracted the interest of researchers in recent decades, FC involving derivatives and integrals Q3 of arbitrary order [1,2]. The fractional representation of the standard diffusion equation has been studied in Refs. [3–6]. Q4 Gorenflo and Mainardi studied the time fractional diffusion equation obtained from a fractional Fick law, the fundamental obtained solution is related to a phenomenon of slow anomalous diffusion [6]. The authors in Ref. [7] studied subdiffusive transport equations in the Caputo and Riemann–Liouville sense. Anomalous diffusion and relaxation behaviors are often described in terms of fractional equations and generalized stochastic equations [8]. Examples for subdiffusion include charge carrier motion in amorphous semiconductors [9], diffusion on fractals [10], nuclear magnetic resonance, glassy materials and transport on fractal geometries [11]. Tadjeran in Ref. [12] obtained temporally and spatially second-order accurate numerical solutions of the fractional order diffusion equations, the authors derived the solutions based on the classical

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Crank–Nicholson method. Based on kernel-based approximation technique, the authors of Ref. [13] proposed an efficient and accurate numerical scheme for solving a backward space–time fractional diffusion problem. Sun in Ref. [14] developed a variable-order fractional diffusion equation, the model characterizes diffusion process in inhomogeneous porous media. In recent papers of Luchko [15–17], the generalized time-fractional diffusion equation with variable coefficients was considered. The author shows the existence and uniqueness of the solution for the initial boundary value problem; the Fourier method was used to construct a formal solution. Other applications of fractional calculus in anomalous diffusion are given in Refs. [18–28].

The Riemann–Liouville and Caputo representations have the disadvantage that their kernel had singularity, this kernel includes memory effects and therefore both definitions cannot accurately describe these effects [29]. In Ref. [30] the authors present the Caputo–Fabrizio fractional derivative, this derivative possesses very interesting properties, for instance, the possibility to describe fluctuations and structures with different scales. The novelty in this operator is that, the derivative has regular kernel, nevertheless, due to these properties some researchers have concluded that this operator can be viewed as filter regulator. Properties and applications of this new fractional derivative are reviewed in detail in the papers [31–35]. Recently Atangana and Baleanu proposed a new kernel based on the Mittag-Leffler function [36–38]. This kernel is nonlocal, nonsingular and has all the benefits of Riemann–Liouville, Caputo and Caputo–Fabrizio fractional derivatives.

The main aim of this work is to obtain analytical solutions for the diffusion equation applying the Atangana–Baleanu fractional derivatives in Caputo and Riemann–Liouville sense, in these representations the dimensionality of the ordinary derivative operator was analyzed in order to define a dimensionally correct fractional derivative operator [39].

The paper is structured as follows, in Section 2 we explain the basic definitions of the fractional calculus, in Section 3 we present the fractional diffusion equation and give conclusions in Section 4.

2. Basic definitions

The Atangana–Baleanu fractional derivative in Caputo sense is defined as follows [36–38]

$$ {}^{ABC}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t \dot{f}(\theta) E_\alpha\Big[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\Big]\,\mathrm{d}\theta, \quad 0<\alpha\le 1 \tag{1}$$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} = {}^{ABC}_{a}\mathcal{D}^{\alpha}_{t}$ is an Atangana–Baleanu fractional derivative in Caputo sense with respect to t, $B(\alpha)$ is a normalization function and has the same properties as in Caputo and Caputo–Fabrizio case.

The Laplace transform of (1) is defined as follows

$$\begin{aligned} \mathscr{L}[{}^{ABC}_{a}\mathcal{D}^{\alpha}_{t} f(t)](s) &= \frac{B(\alpha)}{1-\alpha}\mathscr{L}\Big[\int_a^t \dot{f}(\theta) E_\alpha\Big[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\Big]\mathrm{d}\theta\Big] \\ &= \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha \mathscr{L}[f(t)](s) - s^{\alpha-1}f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \end{aligned} \tag{2}$$

The definition of the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is defined as follows [36–38]

$$ {}^{ABR}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_b^t f(\theta) E_\alpha\Big[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\Big]\,\mathrm{d}\theta, \quad 0<\alpha\le 1 \tag{3}$$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} = {}^{ABR}_{a}\mathcal{D}^{\alpha}_{t}$ is an Atangana–Baleanu fractional derivative in Riemann–Liouville sense with respect to t, $B(\alpha)$ is a normalization function as in the above definition.

The Laplace transform of (3) is defined as follows

$$\begin{aligned} \mathscr{L}[{}^{ABR}_{a}\mathcal{D}^{\alpha}_{t} f(t)](s) &= \frac{B(\alpha)}{1-\alpha}\mathscr{L}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\theta) E_\alpha\Big[-\alpha\frac{(t-\theta)^\alpha}{1-\alpha}\Big]\mathrm{d}\theta\Big] \\ &= \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha \mathscr{L}[f(t)](s)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \end{aligned} \tag{4}$$

Properties of these new fractional derivatives are reviewed in detail in Atangana and Koca [38].

3. Fractional diffusion equation

The local diffusion equation is represented in (5)

$$ D\frac{\partial^2 C(x,t)}{\partial x^2} - \frac{\partial C(x,t)}{\partial t} = E(x,t), \tag{5}$$

Eq. (5) describes the ordinary diffusion, D represents the reciprocal of the time constant of the system or diffusion coefficient.

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