Physica A 461 (2016) 288–298

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Finite current stationary states of random walks on one-dimensional lattices with aperiodic disorder

Hiroshi Miki

Research Institute for Humanity and Nature, 457-4 Motoyama Kamigamo, Kita-ku, Kyoto, 603-8047, Japan

HIGHLIGHTS

- Stationary state of random walk on an aperiodically disordered lattice is investigated.
- Size dependence of velocity and multifractal spectrum of the stationary distribution are examined.
- With a finite bias in the infinite size limit, the stationary state is always extended.
- At a certain finite size scaling behavior changes from a singular or localized to an extended state.

ARTICLE INFO

Article history:
Received 11 April 2016
Available online 2 June 2016

Keywords:
Random walk
Aperiodic disorder
Stationary states
Scaling
Multifractal

ABSTRACT

Stationary states of random walks with finite induced drift velocity on one-dimensional lattices with aperiodic disorder are investigated by scaling analysis. Three aperiodic sequences, the Thue–Morse (TM), the paperfolding (PF), and the Rudin–Shapiro (RS) sequences, are used to construct the aperiodic disorder. These are binary sequences, composed of two symbols A and B, and the ratio of the number of As to that of Bs converges to unity in the infinite sequence length limit, but their effects on diffusional behavior are different. For the TM model, the stationary distribution is extended, as in the case without current, and the drift velocity is independent of the system size. For the PF model and the RS model, as the system size increases, the hierarchical and fractal structure and the localized structure, respectively, are broken by a finite current and changed to an extended distribution if the system size becomes larger than a certain threshold value. Correspondingly, the drift velocity is saturated in a large system while in a small system it decreases as the system size increases.

1. Introduction

A random walk on a lattice, or more generally on a complex network, is a simple stochastic process which describes a classical transport phenomenon in a real space or a step-by-step state change in an abstract state space. Due to and in spite of the simplicity of the process, various and nontrivial properties have been found and analyzed in detail, both mathematically and physically [1,2].

In the presence of disorder, *i.e.*, where the hop probability or rate from one site to another is not uniform, the behavior of the random walk is strongly modified, especially in lower dimensions, not only quantitatively but also qualitatively. When the disorder is random and uncorrelated, various methods have been developed for analysis. Especially, from methods based on renormalization group, many results, some of which are exact, have been obtained [3–6]. One of the most remarkable

E-mail address: miki@chikyu.ac.jp.

http://dx.doi.org/10.1016/j.physa.2016.05.057

results is so-called ultraslow diffusion, where the diffusion is strongly suppressed by disorder, and the averaged mean-square displacement grows extremely slowly, *i.e.* on a log-time scale [7,8]. Correspondingly, the stationary state on a finite lattice is strongly localized.

Systems with aperiodic disorder are also interesting for investigation. An aperiodic disorder is generated by a certain set of deterministic rules but does not have any periodicity. It is this point that distinguishes aperiodic from random uncorrelated disorder. Moreover, aperiodic disorder is considered to be intermediate between uniformity or periodicity and random disorder. Hence, the study of systems with aperiodic disorder is probably a good first step towards understanding systems with more general correlated disorder. Generally, it is difficult to construct an aperiodic disorder with desirable characteristics. Fortunately, for one-dimensional lattice systems various aperiodic disorders can be easily constructed with the help of the aperiodic sequences which have been investigated mathematically. Note that in addition to the theoretical and mathematical interest, systems with aperiodic disorder have been fabricated artificially and investigated experimentally [9]. As expected, some results that are unique for aperiodically disordered systems have been obtained. One of the most remarkable is the appearance of anomalous diffusion, where the mean-square displacement grows slowly — less than linearly with time [10]. Correspondingly, a singular stationary probability distribution with a remarkable hierarchical structure appears [11].

If the hop rates do not satisfy a certain condition (see Eq. (7)), a finite drift velocity is induced, at least in a finite system, and in the stationary state a finite current flows through the lattice. As is well known, a finite current forces the stationary distribution to be extended. Therefore, it is an interesting problem to investigate how the stationary distribution without current will be changed by the presence of a finite current — particularly when the distribution is localized or singular. In the present paper we use scaling analysis to cope with this problem for the cases of lattices with aperiodic disorder. As in our previous study [11], we consider the aperiodic disorders constructed by the Thue–Morse (TM), paperfolding (PF), and Rudin–Shapiro (RS) sequences. These three aperiodic sequences have several common properties: (i) They are binary sequences, which are composed of two types of symbols, A and B. (ii) They are constructed systematically from initial sequences and by iteration of specific substitution rules. (iii) The ratio of the number of As to that of Bs converges to unity in the infinite sequence length limit. Nevertheless, these aperiodic disorders have different effects on the diffusional behavior [10] and correspondingly on the stationary probability distribution [11], since they have difference wandering exponents (see Section 2.2).

We focus on the dependence of the drift velocity and the localization structure of the stationary probability distribution on the system size. In order to characterize the latter, we use multifractal analysis [12], as in our previous study [11]. This approach has been applied to characterize the scaling structure of distributions in various systems, including those of the energy dissipation in turbulence [13,14], the sidebranch structure of dendrites [15], and the quantum localization problem [16], where the localization property of the wavefunction is studied.

The organization of the rest of this paper is as follows: In Section 2, we formulate our model and method for analysis. We describe our one-dimensional random walk, give the expressions of the observables, and introduce the aperiodic sequences from which the disorder is constructed. Then we describe the method of the multifractal analysis for the distribution on a one-dimensional support, the criterion for localization and the finite-size effect. In Section 3, we present our results and a discussion. Section 4 is dedicated to our conclusion and future outlook.

2. Model and method

2.1. Random walk on one-dimensional disordered lattice

Let us consider a random walk on a one-dimensional lattice with only nearest neighbor hopping allowed. This process is described by the master equation:

$$\frac{\partial p_j(t)}{\partial t} = w_{j-1,j}p_{j-1}(t) + w_{j+1,j}p_{j+1}(t) - (w_{j,j-1} + w_{j,j+1})p_j(t), \tag{1}$$

where $p_j(t)$ is the probability for the particle to be on site j at time t and $w_{j,k}$ denotes the hop rate for the particle from site j to k. We impose the periodic boundary condition $p_{j+L} \equiv p_j$ and $w_{j+L,k+L} = w_{j,k}$, where L denotes the system size, the number of sites on the lattice. Interestingly this master equation is known to be equivalent to the transverse-field Ising model [10].

We construct the disorder according to an aperiodic binary sequence, S, composed of two types of symbols, A and B. For example, let us take $S = ABAABAABAB \cdots$. For this sequence, the hop rates are assigned as follows:

$$w_{j,j+1} = 1, \quad \text{for all } j, \tag{2}$$

and

$$w_{j+1,j} = \begin{cases} a, & \text{the } j\text{th symbol of } S \text{ is } A, \\ b, & \text{the } j\text{th symbol of } S \text{ is } B. \end{cases} \tag{3}$$

At least as far as we are concerned with the stationary state, the assignment does not lose generality, since the quantities related to the stationary state, the probability distribution and the drift velocity, are expressed as a function of the ratio $w_{j,j+1}/w_{j+1,j}$ [8].