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On the relationship between the Hurst exponent, the ratio of the mean square successive difference to the variance, and the number of turning points

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HIGHLIGHTS

- fBm, fGn and DfGn are examined.
- Correlations between the Abbe value, the number of turning points, and HE are found.
- HEs can be retrieved based only on the Abbe values.
- Real world data are tested.

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ABSTRACT

The long range dependence of the fractional Brownian motion (fBm), fractional Gaussian noise (fGn), and differentiated fGn (DfGn) is described by the Hurst exponent H. Considering the realizations of these three processes as time series, they might be described by their statistical features, such as half of the ratio of the mean square successive difference to the variance, $\mathcal{A}$, and the number of turning points, T. This paper investigates the relationships between $\mathcal{A}$ and H, and between T and H. It is found numerically that the formulae $\mathcal{A}(H) = ae^{bH}$ in case of fBm, and $\mathcal{A}(H) = a + bH^c$ for fGn and DfGn, describe well the $\mathcal{A}(H)$ relationship. When $T(H)$ is considered, no simple formula is found, and it is empirically found that among polynomials, the fourth and second order description applies best. The most relevant finding is that when plotted in the space of $(\mathcal{A}, T)$, the three process types form separate branches. Hence, it is examined whether $\mathcal{A}$ and T may serve as Hurst exponent indicators. Some real world data (stock market indices, sunspot numbers, chaotic time series) are analyzed for this purpose, and it is found that the H's estimated using the $H(\mathcal{A})$ relations (expressed as inverted $\mathcal{A}(H)$ functions) are consistent with the H's extracted with the well known wavelet approach. This allows to efficiently estimate the Hurst exponent based on fast and easy to compute $\mathcal{A}$ and T, given that the process type: fBm, fGn or DfGn, is correctly classified beforehand. Finally, it is suggested that the $\mathcal{A}(H)$ relation for fGn and DfGn might be an exact (shifted) 3/2 power-law.

1. Introduction

Long-range dependence (LRD) is a feature of a time series that has found applications in a variety of fields, such as condensed matter [1,2], physiology [3], Solar physics [4,5], financial analyses [6,7], and astrophysical phenomena [8,9],

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among others. In general, a process has the LRD property if the autocorrelations $\rho(k)$ decay to zero so slowly that their sum does not converge [10]. More specifically, when

$$\rho(k) \approx c_\rho |k|^{-\delta}, \tag{1}$$

where c_ρ is a positive constant and $0 < \delta < 1$, the process has LRD. The dependence between events (observations) that are far apart diminishes very slowly (slower than $|k|^{-1}$) with increasing k. A stationary process with autocorrelations decaying as in Eq. (1) is called a stationary process with LRD or with long-term memory.

The LRD is usually quantified with the Hurst exponent [11], denoted H. To connect LRD with H, one introduces self-similar processes. A (stochastic) process $x(t)$ is called self-similar with a Hurst exponent H if

$$x(t) \doteq \lambda^{-H} x(\lambda t), \tag{2}$$

where $\doteq$ denotes equality in distribution. If the increments $x(t) - x(t-1)$ are stationary, e.g. a fractional Brownian motion (see further in the text), the autocorrelation function is given by [10]

$$\rho(k) = \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right). \tag{3}$$

A Taylor expansion of $\rho(k)$ from Eq. (3) gives

$$\frac{\rho(k)}{H(2H-1)|k|^{2H-2}} \to 1 \tag{4}$$

for $k \to \infty$. It follows that for $H > 1/2$, the autocorrelation $\rho(k)$ behaves like $|k|^{2-2H}$, thus $x(t)$ has LRD.

The Hurst exponent, introduced by H. E. Hurst in 1951 [11] to model statistically the cycle of Nile floods, is closely related to the concept of a Brownian motion (Bm, also called a random walk or a Wiener process.) [12], for which the consecutive increments are independent, and the standard deviation σ scales with step n as $\sigma \propto n^{1/2}$. Hurst found that in the case of Nile the increments were not independent, but characterized by a power law with an exponent greater than 1/2. This leads to the concept of a fractional Brownian motion (fBm) [12], in which the system possesses the LRD or long-term memory (also called persistency), meaning that its past increments influence the future ones, and the process tends to maintain the increments' sign. For instance, in a persistent process, if some measured quantity attains relatively high values, the system prefers to keep them high. The process is, however, probabilistic [13], and hence at some point the observed quantity will eventually drop to oscillate around some relatively low value. But the process still has long-term memory (being a global feature), therefore it prefers to stay at those low values until the transition occurs stochastically again. The scaling of the standard deviation in such a process is $\sigma \propto n^H$. Finally, an fBm is a non-stationary process.

H can be also smaller than 1/2. In this case, the process is anti-persistent, and it possesses short-term memory, meaning that the observed values of some quantity frequently switch from relatively high to relatively low values (speaking more precisely, the autocorrelations $\rho(k)$ decay fast enough so that their sums converge to a finite value), and there is no preference among the increments. Because the Hurst exponent is also related to the autocorrelation of a time series, i.e. to the rate of its decrease, a persistent process with $H > 1/2$ is sometimes also called correlated, and an anti-persistent one, with $H < 1/2$, is called anti-correlated. Finally, H is bounded to the interval $(0, 1)$. The properties of H can be summarized as follows:

1. $0 < H < 1$,
2. $H = 1/2$ for a Bm (random walk),
3. $H > 1/2$ for a persistent (long-term memory, correlated) process,
4. $H < 1/2$ for an anti-persistent (short-term memory, anti-correlated) process.

Furthermore, the Hurst exponent is related to a fractal dimension of a one-dimensional time series $D \in (1, 2)$ via $D = 2 - H$ [14]. This can be also generalized to processes in higher dimensions d: $D = d + 1 - H$ [15].

The above described scaling is not unique for fBm (a non-stationary process) only. It occurs also in a fractional Gaussian noise (fGn, for which $\rho(k)$ is also in the form given by Eq. (3)), defined by

$$fGn_H(t) = B_H(t+1) - B_H(t), \tag{5}$$

where B_H is an fBm with a Hurst exponent equal to H. The increments of an fBm, forming an fGn, are described by the same H as the fBm itself [16]. Similarly, one defines a differentiated fGn (DfGn) as consecutive increments of an fGn, constructed similarly to Eq. (5), i.e. as

$$DfGn_H(t) = fGn_H(t+1) - fGn_H(t). \tag{6}$$

Both fGn and DfGn are stationary processes.

Fig. 1 shows simulated paths of length 2^{10}, being realizations of fBm and fGn, with $H = 0.2$ and $H = 0.8$, using the method described in Section 2. For the smaller H, the paths are more ragged and the range on the vertical axis is much smaller in the case of fBm due to the reverting behavior of the time series [10].