Physica A 461 (2016) 802–811

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Kinetic and mean field description of Gibrat's law

Giuseppe Toscani

Department of Mathematics, University of Pavia, via Ferrata 1, 27100 Pavia, Italy

HIGHLIGHTS

- A kinetic description of Gibrat's law is presented.
- The possible mean field limits are analyzed.
- The validity of the diffusion limit is rigorously justified.

ARTICLE INFO

Article history:
Received 4 April 2016
Available online 15 June 2016

Keywords:
Kinetic models
Gibrat's law
Linear diffusion equations
Large-time behavior

ABSTRACT

I introduce and analyze a linear kinetic model that describes the evolution of the probability density of the number of firms in a society, in which the microscopic rate of change obeys to the so-called law of proportional effect proposed by Gibrat (1930, 1931). Despite its apparent simplicity, the possible mean field limits of the kinetic model are varied. In some cases, the asymptotic limit can be described by a first-order partial differential equation. In other cases, the mean field equation is a linear diffusion with a non constant diffusion coefficient that can be studied analytically, by virtue of a transformation of variables recently utilized in Iagar and Sánchez (2013) to study the heat equation in a nonhomogeneous medium with critical density. In this case, it is shown that the large-time behavior of the solution is represented, for a large class of initial data, by a lognormal distribution with constant mean value and variance increasing exponentially in time at a precise rate.

1. Introduction

The agent-based models represent a broad class of models which have been recently introduced to describe various phenomena of economic dynamics in a more realistic way [1–3]. Much has been done in the past fifteen years by providing new models that can reproduce various features of financial markets, like volatility clustering and fat tails of returns [4–10]. This relatively new research field borrows several methods and tools from classical statistical mechanics, where emerging complex behavior arises from relatively simple rules as a consequence of microscopic interactions among a large number of agents [1,2].

Starting from the microscopic dynamics, kinetic models can be derived by resorting to well-known tools of classical kinetic theory of gases [9,11–14], where kinetic econo-physics has been treated in the framework of Boltzmann-like equation for Maxwell-type molecules [15,2]. In contrast with microscopic dynamics, where the large-time behavior of the system can be often studied only empirically through computer simulations, kinetic models based on integro-differential and/or partial differential equations allow us to derive analytically general information on the model and its asymptotic behavior.

Among the various interactions models that can be studied by this powerful methodology, one of the simplest ones is certainly Gibrat's law for firm growth [16–19]. Gibrat formulated the law of proportionate effect for growth rate to justify

E-mail address: giuseppe.toscani@unipv.it.

http://dx.doi.org/10.1016/j.physa.2016.06.063
0378-4371/

the observed distributed distribution of firms. The law of proportionate effect states that the expected increment of a firm's size in a fixed period of time is proportional to the size of the firm at the beginning of the period. Denoting by $x(\tau)$ the size of a firm at a time $\tau \geq 0$, the postulate is expressed as

$$x(\tau + 1) = x(\tau) + \eta(\tau)x(\tau), \tag{1.1}$$

where $\eta(\tau)$ is a random number independent of $x(\tau)$, and $\eta(\tau)$ is independent of $\eta(\tau + k)$ for any natural number k, and there are no interactions between firms.

After a sufficiently long sequence of increments, since Gibrat's law implies that

$$x(n) = x(0)(1 + \eta(1))(1 + \eta(2)) \cdots (1 + \eta(n)),$$

$\log x(n)$ follows a random walk. Therefore, the growth rate predicted by Gibrat's law is lognormally distributed with mean and variance linked to the mean and variance of $\eta(\cdot)$. The validity of Gibrat's law has been investigated by many authors, often with a critical viewpoint [20,21].

Despite its simplicity, or maybe in reason of this, continuous kinetic models based on Gibrat's law seem to have not yet been studied. Of course, the rate of change expressed by (1.1) appears as part of the microscopic binary interaction between agents in kinetic models for wealth distribution, like in the model proposed by the author with Cordier and Pareschi [9], where the term $\eta(\tau)x(\tau)$ plays the role of the risk in an economic trade in which $x(\tau)$ denotes the wealth of the trader at time τ. Also, a law similar to (1.1) appears in the pure gambling model studied in Ref. [22] to investigate the possibility to generate Pareto tails by conservative-in-the-mean interactions.

In this paper we aim to study both kinetic and mean field models generated by interactions of type (1.1). Depending on the properties of the random variable η, various limiting behaviors appear, that, while maintaining the main properties (conservation of the mean number of firms, growth of higher moments, etc.) exhibit completely different asymptotic behaviors. Among others, we will show that Gibrat's law can be described in terms of the mean field equation

$$\frac{\partial u}{\partial t} = \frac{\sigma}{2} \frac{\partial^2}{\partial x^2}(x^2 u), \tag{1.2}$$

where $\sigma > 0$ is a fixed constant. Eq. (1.2) contains in fact the main effects of Gibrat's law (1.1) when the random variable η produces small symmetric effects. The linear diffusion equation (1.2) allows to describe the evolution in time of the density $u = u(x, t)$ of the size $x \geq 0$ of firms, given their distribution $u_0(x)$ at time $t = 0$, as well as its asymptotic behavior. It is noticeable that the solution of Eq. (1.2) can be described analytically by resorting to a transformation of variables first introduced recently by Iagar and Sánchez [23] in connection with the study of the heat equation in a nonhomogeneous medium with critical density. This transformation allows to establish a deep connection of Eq. (1.2) with the standard heat equation.

Owing to this connection, we will show that Eq. (1.2) possesses a source-type solution which is nothing but a lognormal density of unit mean and variance exponentially growing in time at a rate $2t$, departing from a Dirac delta function located at $x = 1$. This source-type solution plays the role of the fundamental solution in the heat equation to build up any other solution to Eq. (1.2).

2. The model

Let us consider a system composed of a huge number of agents which are identified in terms of a certain characteristic, which can be modified by some universal interaction rule. If this characteristic is measured by a nonnegative number x, the aim of a kinetic model is to provide a continuous description for the evolution in time, denoted by τ, of the density function $f(x, \tau)$ of the x-variable consequent to interactions [1,2].

Let us assume that the population of agents coincide with the list of firms. Then the precise meaning of the density f is the following. Given the list of firms to study, and a domain $D \subseteq \mathbb{R}_+$, the integral

$$\int_D f(x, \tau)\, dx$$

represents the percentage of firms with size included in D at time $\tau \geq 0$. A natural assumption is to normalize to one the density function, that is

$$\int_{\mathbb{R}_+} f(x, \tau)\, dx = 1$$

for any time $\tau \geq 0$. According to Gibrat's postulate (1.1), we will assume that the microscopic variation of the firm size is due to interactions with the external background, and it is proportional to the size itself. Consequently, given a firm of size x, its post-interaction size is given by

$$x^* = x + \eta x, \tag{2.3}$$

where the random quantity ηx represents the change in size of the firm, proportional to the pre-interaction size x, generated by the presence of the background. We will assume that the random variable η, takes values in a bounded set limited