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Fractional randomness

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HIGHLIGHTS

- How are probability distributions altered when a fractional operator is applied. We provide a statistical definition of fractional randomness based on uniform as well as discrete distributions. Practically, it implies that a distribution's granularity defined by a fractional kernel may have properties that differ due to the fractional index used and the fractional calculus applied to define it.
- Do such fractional distributions remain probability distributions in the conventional statistical sense or not. For increasing and decreasing functions that are right continuous we define fractional probability distributions. Practically, we show that a random variable with a known and conventional density function may not, in a fractional context, define a complete probability distribution, i.e. a distribution with positive probabilities, summing to one over the space they are defined by.
- We point out that fractional distributions are defined relative to both their granularity and the functional form of the distribution. Examples and applications are used to motivate a statistical approach to fractional calculus that leads to the definition of conventional fractional probability distributions. Some of the pitfalls in interpreting fractional distributions as probability distributions are also indicated.

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ABSTRACT

The premise of this paper is that a fractional probability distribution is based on fractional operators and the fractional (Hurst) index used that alters the classical setting of random variables. For example, a random variable defined by its density function might not have a fractional density function defined in its conventional sense. Practically, it implies that a distribution's granularity defined by a fractional kernel may have properties that differ due to the fractional index used and the fractional calculus applied to define it. The purpose of this paper is to consider an application of fractional calculus to define the fractional density function of a random variable. In addition, we provide and prove a number of results, defining the functional forms of these distributions as well as their existence. In particular, we define fractional probability distributions for increasing and decreasing functions that are right continuous. Examples are used to motivate the usefulness of a statistical approach to fractional calculus and its application to economic and financial problems. In conclusion, this paper is a preliminary attempt to construct statistical fractional models. Due to the breadth and the extent of such problems, this paper may be considered as an initial attempt to do so.

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1. Introduction

The premise of this paper is that a fractional probability distribution is based on operators that alter the underlying and conventional assumptions regarding random variables, their properties and their functional density functions. A fractional random variable is inherently defined by its granularity and the fractional index that defines it. These result in properties that need not be conventional in a statistical sense. Practically, it implies that a random variable with a known and conventional density function may not, in a fractional context, define a complete probability distribution, i.e. a distribution with positive probabilities, summing to one over the space they are defined by. Such distributions may then have statistical properties that differ and functional forms that depend on the fractional index. In this case, a probability distribution model is necessarily defined by the data and its granularity as well as by the statistical treatment of such data. For example, financial day-data, intra-day or high frequency data, although all measuring prices, the statistical properties of traded stocks, currencies and financial assets, may require that their granularity be accounted for when modeling and estimating their properties. The statistical fractional approach we consider in this paper differs from the extensive literature based on financial models assuming that Fractional Brownian Motion "drives" models' risks. Rather, we provide a preliminary approach to assess a fractional random probability distribution, its existence in a statistical conventional sense as well as the necessary conditions for such distributions to be concurrent with the properties of conventional statistical distributions.

Statistical fractional models have proved useful for the study of auto-correlated time series that have properties that did not conform to traditional and statistically independent regressions ARIMA models. During the eighties, the study of financial day data time series revealed non-linear statistical properties, including mainly autoregressive heteroscedasticity. Non-linear models such as ARCH, GARCH suggested by Bollerslev [1], Engle [2] to estimate financial volatility were provided (see also Refs. [3–8] and many others). Time series analyses of various granularity (such as intra-day data) also revealed a new wealth of properties such as day seasonal hetero-skedasticity which cannot be fully covered by ARCH and GARCH models (for example, see Refs. [9–12]). In finance, Mandelbrot and Van Ness [13] as well as subsequent R/S studies of FX (Foreign Exchange) and other financial data were shown to exhibit an autocorrelation, and thereby, a long run dependence defined in terms of the Hurst fractional index (see Refs. [14–20]). The effects of such an index on volatility models were translated into a fractional volatility in Brownian Motion models (also called Fractional Brownian Motion, see also Refs. [21–27] and others). Other studies have also pointed out to the existence of fractional properties observed in the study of high-density foreign exchange (FX) data leading at their limit to power law distributions (see Ref. [9]). In an autocorrelation study with high-density data, Dacorogna et al. [12] have also shown that the absolute values of price changes were behaving as a "fractional noise" unlike the absolute price changes expected from GARCH models where past volatility effects decline hyperbolically with time. Based on ARCH and GARCH models, derived models based on ARFIMA regressions were subsequently used to account for the underlying fractional volatility of time series. Finally, in Physics, similar observations were made, coined non-locality, providing an intertemporal statistical relationship.

In this paper, we believe that the definitions of fractional statistical probability distributions may provide an additional and complementary approach to estimate the fractional distributions' index (albeit, these are mostly complex numerical problems) as well as provide a more explicit definition of models defined relatively to their granularity and the fractional operators they use to assess their properties. To do so we shall use a Riemann–Liouville fractional calculus (although we shall refer as well to other operators). References to fractional calculus and to various statistical elements and their applications to stochastic processes abound, including for example Laskin [28], Jumarie [29], Baleanu, Diethhlem, Scallas and Trujillo, [30], Almeida, Pooseh, Delfim and Torres [31] and many others broadly available in academic papers and books we shall refer to subsequently.

The approach of this paper differs from fractional diffusion models in the sense that we focus our attention to the definition of a fractional randomness and the existence of a fractional probability distribution in its conventional sense. For simplicity, we have focused our attention to uniform random variables as they underlie many distributions and processes and attempt to explain within a fractional calculus framework the meaning of an "incomplete randomness". In particular the effects of granularity that contribute to the existence or the non-existence of a complete fractional randomness represented by uniform distributions. We also consider specific cases including discrete probability distributions to emphasize the fact that data is mostly discrete rather than continuous and ought to be interpreted as such. Further research is of course required to study the statistical properties of fractional probability distributions across a broader range of financial indexes and assess their implications to the underlying data and models they are based on.

2. A cursory review of fractional calculus and operators

The origins of fractional calculus parallel early developments of calculus. Leibniz linear product rule: $\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x$, extended to higher order and integer derivatives such as (see for example, Refs. [32,33], for a review):

$$\mathrm{d}^k(xy) = \mathrm{d}^k x \mathrm{d}^0 y + \frac{k}{1!}\mathrm{d}^{k-1}x\mathrm{d}^1 y + \frac{k(k-1)}{2!}\mathrm{d}^{k-2}x\mathrm{d}^2 y + \cdots . \tag{2.1}$$