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On the excursions of drifted Brownian motion and the successive passage times of Brownian motion

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HIGHLIGHTS

- It is studied the law of the excursions of Brownian motion with drift.
- We find the distribution of the nth passage time of Brownian motion through a straight line.
- We extend the results to space–time transformations of Brownian motion.

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ABSTRACT

By using the law of the excursions of Brownian motion with drift, we find the distribution of the nth passage time of Brownian motion through a straight line $S(t) = a + bt$. In the special case when $b = 0$, we extend the result to a space–time transformation of Brownian motion.

1. Introduction

We consider a drifted Brownian motion of the form

$$X(t) = x + \mu t + B_t, \quad t \geq 0, \tag{1.1}$$

where $x, \mu \in \mathbb{R}$ and B_t is standard Brownian motion (BM). When $X(t)$ is entirely positive or entirely negative on the time interval (a, b), it is said that it is an excursion of drifted BM; this means that B_t remains above or below the straight line $-x - \mu t$, for all $t \in (a, b)$. Excursions of drifted BM have interesting applications in Biology, Economics, and other applied sciences. As an example in Economics, if we admit that the time evolution of the gross domestic product is described by a BM with drift μ, starting from x, then the downward and upward movement of it around its long-term growth trend (i.e. the straight line $x + \mu t$), gives rise to an economic cycle. These fluctuations typically involve shifts over time between periods of relatively rapid economic growth (expansions or booms), and periods of relative stagnation or decline (contractions or recessions) (see e.g. Ref. [1]). Excursions of drifted BM are also related to the last passage time of BM through a linear boundary; actually, last passage times of continuous martingales play an important role in Finance, for instance, in models of default risk (see e.g. Refs. [2,3]).

When the drift μ is zero, $X(t)$ becomes BM and it is well-known that the excursions of BM have the arcsine law, namely the probability that BM has no zeros in the time interval (a, b) is given by $\frac{2}{\pi}$ arcsin $\sqrt{a/b}$ (see e.g. Ref. [4]). By using Salminen's formula for the last passage time of BM through a linear boundary (see Ref. [5]), we find the law of the excursions of drifted

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BM, namely the probability that $X(t) = x + \mu t + B_t$ has no zeros in the interval (a, b). From this, we derive the distribution of the nth passage time of BM through the linear boundary $S(t) = a + bt,\ t \geq 0$.

We recall that the first-passage time of BM through $S(t)$, when starting from x, is defined by $\tau_1(x) = \inf\{t > 0 : x + B_t = a + bt\}$ and the Bachelier–Levy formula holds:

$$P(\tau_1(x) \leq t) = 1 - \Phi((a-x)/\sqrt{t} + b\sqrt{t}) + \exp(-2b(a-x))\Phi(b\sqrt{t} - (a-x)/\sqrt{t}),$$

where $\Phi(y) = \int_{-\infty}^{y} \phi(z)\mathrm{d}z$, with $\phi(z) = \mathrm{e}^{-z^2/2}/\sqrt{2\pi}$, is the cumulative distribution function of the standard Gaussian variable. If $(a-x)b > 0$, then $P(\tau_1(x) < \infty) = \mathrm{e}^{-2b(a-x)}$, whereas, if $(a-x)b \leq 0,\ \tau_1(x)$ is finite with probability one and it has the following Inverse Gaussian density, which is non-defective (see e.g. Ref. [6]):

$$f_{\tau_1}(t) = f_{\tau_1}(t|x) = \frac{\mathrm{d}}{\mathrm{d}t}P(\tau_1(x) \leq t) = \frac{|a-x|}{t^{3/2}}\,\phi\left(\frac{a+bt-x}{\sqrt{t}}\right), \quad t > 0; \tag{1.2}$$

moreover, if $b \neq 0$, the expectation of $\tau_1(x)$ is finite, being $E(\tau_1(x)) = \frac{|a-x|}{|b|}$.

The second-passage time of BM through $S(t)$, when starting from x, is defined by $\tau_2(x) = \inf\{t > \tau_1(x) : x + B_t = a + bt\}$, and generally, for $n \geq 1,\ \tau_n(x) = \inf\{t > \tau_{n-1}(x) : x + B_t = a + bt\}$ denotes the nth passage time of BM through $S(t)$. Our aim is to study its distribution.

The paper is organized as follows: in Section 2 we will find explicitly the distribution of the nth passage time of BM, in Section 3, we will extend the result to space–time transformations of BM, in the special case when $b = 0$.

2. The *n*th passage time of Brownian motion

In this section we suppose that $b \leq 0$ and $x < a$, or $b \geq 0$ and $x > a$, so that $P(\tau_1(x) < \infty) = 1$. First, for fixed $t > 0$, we consider the last-passage-time prior to t of BM, starting from x, through the boundary $S(t) = a + bt$, that is:

$$\lambda_S^t = \sup\{0 \leq u \leq t : x + B_u = S(u)\}. \tag{2.1}$$

The distribution of λ_S^t can be expressed in terms of the first-passage-time distribution of BM through the time-reversed boundary $\widehat{S}(u) = S(t-u)$ (see Ref. [5]); in particular, we can derive from Ref. [5] the following formula for the probability density, say $\psi_t(u)$, of λ_S^t:

$$\psi_t(u) = \frac{\mathrm{d}}{\mathrm{d}u}P(\lambda_S^t \leq u) = \frac{1}{\sqrt{2\pi u}}\exp(-b^2u/2)\int_{-\infty}^{+\infty} \nu_{x-a}(t-u, \widehat{S})\mathrm{d}x, \quad u \leq t \tag{2.2}$$

where:

$$\nu_x(v, \widehat{S}) = \exp\{-b(x-bt) - b^2v/2\}\frac{|x-bt|}{\sqrt{2\pi v^3}}\exp\{-(x-bt)^2/2v\}. \tag{2.3}$$

Then, the following explicit formula is obtained, by calculation:

Lemma 2.1. *The probability density of λ_S^t is explicitly given by:*

$$\psi_t(u) = \frac{\mathrm{e}^{-\frac{b^2}{2}u}}{\pi\sqrt{u(t-u)}}\left[\mathrm{e}^{-\frac{b^2}{2}(t-u)} + \frac{b}{2}\sqrt{2\pi(t-u)}\left(2\Phi(b\sqrt{t-u}) - 1\right)\right], \quad 0 < u < t \tag{2.4}$$

where $\Phi(y) = \int_{-\infty}^{y} \phi(z)\mathrm{d}z$ is the distribution function of the standard Gaussian variable. Notice that ψ_t is independent of a. In particular, if $b = 0$, one gets:

$$\psi_t(u) = \frac{1}{\pi\sqrt{u(t-u)}}, \quad 0 < u < t, \tag{2.5}$$

that is, the arc-sine law with support in $(0, t)$.

Proof. By using (2.2) and (2.3), we obtain:

$$\psi_t(u) = \frac{\mathrm{e}^{-b^2u/2}}{\sqrt{2\pi u}} \cdot \frac{\mathrm{e}^{-b^2(t-u)/2}}{t-u} \cdot J(u), \tag{2.6}$$

where

$$J(u) = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-by}|y|}{2\pi\sqrt{t-u}}\mathrm{e}^{-y^2/2(t-u)}\mathrm{d}y.$$