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On some properties of reflected skew Brownian motions and applications to dispersion in heterogeneous media

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ABSTRACT

Motivated by the close connection between the skew Brownian motion and the random particle motion in heterogeneous media, we investigate the reflected skew Brownian motion and try to find out its relationship with the corresponding dispersion problem when there exists a reflecting boundary. Through the use of the knowledge of stochastic analysis, we provide some basic properties of reflected skew Brownian motions, including the transition density, the Laplace transform of the first passage time, and some related results. A simple method to generate the sample path is also proposed. At the end of this paper, we reveal the strong relationship between the reflected skew Brownian motion and the solute dispersion in the presence of a sharp interface and a reflecting boundary.

1. Introduction

Since the concept of skew Brownian motion was first put forward by Itô and McKean [1], scholars have gradually discovered its wide applications in natural sciences. The most typical application of this process as well as its extensions (called the skew diffusions) is to model the random path of particles dispersing in heterogeneous media in physical experiments. In Ref. [2], such a random path was described by a rescaled α-skew Brownian motion where α is determined by the diffusion coefficients on both sides of the interface. A detailed mathematical analysis of multi-skewed Brownian motions was made by Ramirez [3], where the asymptotic behavior of the underlying process of the particle in a two-dimensional periodic layered medium was investigated. Lejay and Pichot [4] proposed a new Monte Carlo method to simulate the path of the diffusive solute in discontinuous media by reducing the process to a skew Brownian motion; Ramirez et al. [5] analyzed the skew diffusion which corresponds to the equations governing solute concentration. Among other things, the authors concluded that a Feller process with concrete speed and scale densities is associated with the particle path in a layered medium with two reflecting boundaries, but no further properties were provided about this process. Besides, to the best of our knowledge, this is the only published paper which takes into account the reflecting boundaries. More works involving the application of skew diffusions on physical experiments of diffusive transport include [6–12] and so on. For the related mathematical theory and their applications in other fields like geophysics, biology, ecology and finance, please refer to Walsh [13], Harrison and Shepp [14], Le Gall [15], Engelbert and Schmidt [16], Bass and Chen [17], Lejay [18], Ramirez, Thomann and Waymire [12], Cai and Yang [19] and references therein.

Inspired by the aforementioned literature on physical dispersion, in this paper, we explore whether the particle motion in a layered medium with one reflecting boundary can be described by the reflected skew Brownian motion. As the first step,

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we investigate the skew Brownian motion with state space $[0, \infty)$ which is reflected at 0. This process $X = \{X_t : t \geq 0\}$ satisfies the following stochastic differential equation (SDE)

$$dX_t = dB_t + (2\alpha - 1)dL_t^X(\beta) + dL_t^X(0), \quad 0 < \alpha < 1, \ \beta > 0, \tag{1}$$

where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t : t \geq 0\}, \mathbf{P})$ with the filtration $\{\mathscr{F}_t : t \geq 0\}$ satisfying the usual conditions, and $L^X(x) = \{L_t^X(x) : t \geq 0\}$ denotes the symmetric local time of the process X at the point $x \geq 0$. The strong existence and pathwise uniqueness of solutions to (1) can be established by Le Gall [15]. In addition, the parameter α results in a special "skew" sample property of X: when X touches the "interface" β, it reflects upwards with probability α and downwards with probability $1-\alpha$. Particularly, if α takes value $1/2$, the process will degenerate to the reflected Brownian motion; if we remove the "reflecting term" $dL_t(0)$, X will become the skew Brownian motion with skew parameter α. However, compared with the vast literature on skew Brownian motions, the basic properties of this process are still unknown and need further investigation.

To begin with, we introduce the physical background of skew Brownian motions, thereby leading to the aim of our paper: probing into the probabilistic properties of the reflected skew Brownian motion and its connection with solute transport in layered media with one reflecting boundary. The study of the properties are of great importance in both themselves and their applications (see, e.g., Refs. [20,21] for the use of the transition density and first passage time in parameter estimation and simulation, McKenzie, Lewis and Merrill [22] in animal movement models, Appuhamillage et al. [10], Hathhorn and Charbeneau [23] and Appuhamillage and Sheldon [11] in physical dispersion problems). The transition density is considered in the first place. We derive the Laplace transform of the transition density from a Feynman–Kac-type formula for X by solving a differential equation, and consequently we get the explicit transition density in light of the inverse Laplace formula.[1] A stochastic representation of the transition densities for reflected skew diffusions is also obtained. After that, we study the first passage time problems over a constant and a random jump boundary, and give some corollaries involving expected passage times and occupation times. The main tool we use is the Itô-Tanaka formula and the strong Markov property of X. Later, an acceptance–rejection algorithm is presented to simulate the sample path.

The specific application to the dispersion problem in heterogeneous media is discussed in the end. It involves establishing the relationship between the reflected skew Brownian motion and the random particle motion in the presence of a sharp interface and a reflecting boundary. In essence, their relationship is determined by their corresponding transition densities. Additionally, we deal with the solute breakthrough time and analyze an example concerning the breakthrough time densities.

The rest of the paper is organized as follows. Section 2 summarizes the physical background of the skew Brownian motion based on which the aim of our work is raised. Section 3 exhibits the transition density for the reflected skew Brownian motion. The first passage time problems are treated in Section 4. In Section 5, we show how to simulate the reflected skew Brownian motion through the analysis of its transition density. The connection with the solute dispersion in heterogeneous media is illuminated in the last section.

2. Physical background

The skew Brownian motion $B^{(\alpha,\delta)} = \{B_t^{(\alpha,\delta)} : t \geq 0\}$ meets the SDE

$$dB_t^{(\alpha,\delta)} = dB_t + (2\alpha - 1)dL_t^{B^{(\alpha,\delta)}}(\delta), \quad 0 < \alpha < 1,$$

where $L^{B^{(\alpha,\delta)}}(x) = \{L_t^{B^{(\alpha,\delta)}}(x) : t \geq 0\}$ stands for the symmetric local time of $B^{(\alpha,\delta)}$ in the sense that

$$L_t^{B^{(\alpha,\delta)}}(x) = \lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|B_s^{(\alpha,\delta)} - x| \leq \epsilon\}} ds.$$

At the moment of reaching δ, the process reflects upwards and downwards with probabilities α and $1 - \alpha$, respectively. Owing to this interesting path property, the skew Brownian motion is strongly connected with the solute concentration dispersing in heterogeneous media with sharp interfaces, the theory of which has been studied in, for example, Refs. [5,9,2,3,10,12]. More precisely, consider the simple case of random particle motion in the heterogeneous medium with one interface. Suppose the molecular diffusion coefficient D has the form

$$D(y) = D^- \mathbb{1}_{[-\infty,0]}(y) + D^+ \mathbb{1}_{(0,\infty)}(y),$$

and denote

$$\alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}}, \tag{2}$$

[1] Unlike the case of skew Brownian motions, there are no excursion expressions in series forms for their reflected counterparts, which indicates that the reflection principle used to derive the transition density for skew Brownian motions (see Ref. [13]) becomes ineffective here.