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Dimers on the $3^3.4^2$ lattice

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HIGHLIGHTS

- Obtain explicit expression of the number of close-packed dimers of the $3^3.4^2$ lattice with cylindrical boundary condition.
- Obtain the dimer entropy of $3^3.4^2$ lattice with cylindrical and toroidal boundary conditions.
- Prove that the entropy of $3^3.4^2$ lattice is the same for cylindrical and toroidal boundary conditions.

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ABSTRACT

In this work, we obtain explicit expression of the number of close-packed dimers (perfect matchings) of the $3^3.4^2$ lattice with cylindrical boundary condition. Particularly, we show that the entropy of $3^3.4^2$ lattice is the same for cylindrical and toroidal boundary conditions.

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1. Introduction

A central problem in statistical physics and combinatorial mathematics is the enumeration of close-packed dimers, often referred to as perfect matchings in the mathematical literature, on lattices which mimics the adsorption of diatomic molecules on a surface [1]. In 1961, Kasteleyn [2] found a formula for the number of close-packed dimers of an $m \times n$ quadratic lattice graph with both free and toroidal boundary conditions. Temperley and Fisher [3] used a different method and arrived at the same result at nearly the same time. Both lines of calculation showed that the logarithm of the number of close-packed dimers, divided by $mn/2$, converges to $2c/\pi \approx 0.5831$ as $m, n \to \infty$, where c is Catalan's constant. This limit is called the entropy of the quadratic lattice graph and the corresponding problem was called the dimer problem by the statistical physicists, where the entropy has a factor of Boltzmann's constant and in this paper the Boltzmann factor will be set equal to one. In 1992, Elkies et al. [4] studied the enumeration of close-packed dimers of regions called Aztec diamonds, and showed that the entropy equals $\log 2/2 \approx 0.35$. The problem involving enumeration of close-packed dimers of another type of quadratic lattices with different boundary conditions was studied by Sachs and Zeritz [5] and a different entropy was obtained. These results showed that the entropy of the quadratic lattice can generally be strongly dependent on their boundary conditions. It should be pointed out that the dimer model on the hexagonal (Kasteleyn or brick) lattice has a "frozen" ground state, which sort of resembles the ground state of the ferromagnetic six-vertex model. It has been shown that the entropy of the six-vertex model does depend on the boundary conditions [6].

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Kenyon, Okounkov, and Sheffield [7] considered the problem of enumerating close-packed dimers of the doubly period bipartite graph on a torus. They proved that the number of close-packed dimers of the doubly period plane bipartite graph G can be expressed in terms of four determinants and they expressed the entropy of G as a double integral.

The exact solution of the dimer problem was obtained for many lattices such as the quadratic lattice [8,2], 8.8.4 lattice [9], hexagonal lattice, triangular lattice, Kagome lattice, 3-12-12 lattice, union Jack lattice, etc. with toroidal boundary condition [10]. The exact solution of the dimer problem has been extended to the cylindrical condition [11,12]. Wu and Wang [13] obtained the exact result on the enumeration of close-packed dimers on a finite Kagome lattice with general asymmetric dimer weights under the cylindrical boundary condition. The result by Wu and Wang implies that the Kagome lattices with the cylindrical and toroidal boundary conditions have the same entropy. This phenomenon also took place for some other lattices with the cylindrical and toroidal boundary conditions [14,15], and cylindrical, toroidal and Klein bottle [16].

In this paper, we consider the $3^3.4^2$ lattice. We obtain explicit expressions of the number of close-packed dimers with cylindrical boundary condition and entropy for this lattice with cylindrical and toroidal boundary conditions. Our results imply that the $3^3.4^2$ lattices with cylindrical and toroidal boundary conditions have the same entropy.

2. Pfaffians

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. An even cycle C of G is said to be nice if the subgraph $G - C$ (obtained from G by deleting all vertices of C) has a close-packed dimer. Let D be an orientation of G. An even cycle C is said to be oddly oriented in D if it has an odd number of edges directed in either direction of C. Otherwise C is evenly oriented. We say that an orientation D is a Pfaffian orientation of G if every nice cycle with even length of G is oddly oriented in D. A graph is a Pfaffian graph if it has a Pfaffian orientation. There are some results characterizing the Pfaffian graphs. Little [17] gave a well known result that if a bipartite graph G contains no subdivision of $K_{3,3}$, then G has a Pfaffian orientation. Fischer and Little [18,19] gave a characterization of Pfaffian near bipartite graphs. McCuaig [20], and McCuaig, Robertson et al. [21], and Robertson, Seymour et al. [22] found a polynomial-time algorithm to show whether a bipartite graph has a Pfaffian orientation. Norine and Thomas [23] exhibited an infinite family of minimal non-Pfaffian graphs with respect to the matching minor. For a recent survey of Pfaffian orientations of graphs, please see Thomas [24].

The Pfaffian method enumerating close-packed dimers of plane graphs was independently discovered by Kasteleyn [2], Fisher [8], and Temperley [3]. For a planar graph embedded in the plane, the method produces a skew symmetric matrix A such that the number of close-packed dimers of G can be expressed by the Pfaffian of the matrix A. Alternatively, the Pfaffian can be replaced by the square root of the determinant of A. By using this method, Kasteleyn [2], Fisher [8] and Temperley [3] solved independently a famous problem on enumerating close-packed dimers of an $m \times n$ quadratic lattice graph in statistical physics-Dimer problem. Kasteleyn also extended his approach to toroidal grids in Refs. [2,25,26] with this Pfaffian method. Given a simple graph G with n vertices, let G^e be an arbitrary orientation of G. The skew adjacency matrix of G^e, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (a_{ij})_{n\times n},$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an arc of } G^e, \\ -1 & \text{if } (v_j, v_i) \text{ is an arc of } G^e, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $A(G^e)$ is a skew symmetric matrix. Throughout this paper, we denote by $M(G)$ the number of close-packed dimers of a graph G.

Lemma 2.1 (*Lovaśz et al. [27]*). *If G^e is a Pfaffian orientation of a graph G, then*

$$M(G) = \sqrt{\det(A(G^e))},$$

where $A(G^e)$ is the skew adjacency matrix of G^e.

Lemma 2.2 (*Lovaśz et al. [27]*). *If a connected plane graph G has an orientation G^e such that every boundary face – except possibly the infinite face – has an odd number of edges oriented clockwise, then in every cycle the number of edges oriented clockwise is of opposite parity to the number of vertices of G^e inside the cycle. Consequently, G^e is a Pfaffian orientation.*

3. The number of close-packed dimers and entropies

The $3^3.4^2$ lattice $S^t(2m, 2n)$ with toroidal boundary condition can be constructed by starting with a $2m \times 2n$ square lattice and adding a diagonal edge connecting the vertices in, say, the upper left to the lower right corners of each square in every other row as shown in Fig. 1(a), where $a_1 = b_1, a_{2m} = b_1^*, a_1^* = b_{2n}, a_{2m}^* = b_{2n}^*$, and $(a_1, a_1^*), (a_2, a_2^*), \ldots, (a_{2m}, a_{2m}^*), (b_1, b_1^*), (b_2, b_2^*), \ldots, (b_{2n}, b_{2n}^*), (a_1, a_2^*), (a_3, a_4^*), \ldots, (a_{2m-1}, a_{2m}^*)$ are edges in $S^t(2m, 2n)$. If we delete the

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