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Variational approach and deformed derivatives

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HIGHLIGHTS

- The calculus of variations was extended for systems containing deformed derivatives.
- The extension was carried out in the context of nonextensive statistics.
- The classical Euler–Lagrange equations and the Hamiltonian formalism were extended.
- We consider complex systems and the mapping into the fractal continuum.
- Systems described by nonlinear equations/position-dependent mass were analyzed.

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ABSTRACT

Recently, we have demonstrated that there exists a possible relationship between q-deformed algebras in two different contexts of Statistical Mechanics, namely, the Tsallis' framework and the Kaniadakis' scenario, with a local form of fractional-derivative operators for fractal media, the so-called Hausdorff derivatives, mapped into a continuous medium with a fractal measure. Here, in this paper, we present an extension of the traditional calculus of variations for systems containing deformed-derivatives embedded into the Lagrangian and the Lagrangian densities for classical and field systems. The results extend the classical Euler–Lagrange equations and the Hamiltonian formalism. The resulting dynamical equations seem to be compatible with those found in the literature, specially with mass-dependent and with nonlinear equations for systems in classical and quantum mechanics. Examples are presented to illustrate applications of the formulation. Also, the conserved Noether current is worked out.

1. Introduction

The minimum action principle implies that, by minimizing some action variables or functionals, we can obtain the dynamical equations that describe physical phenomena. The formalism related is known as the calculus of variations. But, the classical variational calculus has a major difficultly in dealing with nonconservative systems.

Here, we claim that the calculus of variations with deformed-derivatives [1] embedded into a Lagrangian or a Lagrangian density is adequate to study both, conservative and nonconservative classical and field systems in order to obtain a version of the respective Euler–Lagrange equations (E–L), combining both cases. Also, systems with position-dependent mass and nonlinearities can be studied and the E–L obtained.

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The applications of the formalism presented here may include classical and quantum mechanics, field theory, complex systems and so on. We are not talking about including classical definitions of fractional derivatives nor including operators of integer order acting on a d-dimensional space but we consider a mapping from a fractal coarse-grained (fractal porous) space, which is essentially discontinuous in the embedding Euclidean space [2,3] to a continuous one. Also, in the construction of the actions to obtain the Euler–Lagrange equations (ELE), we have used different definitions for the actions and for the deformed-derivatives and integrals.

In Ref. [4], we adopt the viewpoint to suitably treat nonconservative systems is through Fractional Calculus (FC), since it can be shown that, for example, a friction force has its form stemming from a Lagrangian that contains a term proportional to the fractional derivative, which may be a derivative of any non-integer order [5]. Parallel to the standard FC, there is some kind of the local fractional calculus with certain definitions called local fractional derivatives, for example, the works of Refs. [6–12]. Here we are interested in the related approaches with Hausdorff derivative [2,6] and the conformable derivative [13]. We think that the most appropriate name of those formalism are deformed-derivatives or even metric or topological derivatives. All of the mentioned approaches seem to be applicable to power-law phenomena. Also, for description of complex systems, the q-calculus, in a non-extensive statistic context, has its formal development based on the definition of deformed expressions for the logarithm and exponential [14], namely, the q-logarithm and the q-exponential. In this context, an interesting algebra emerges and the formalism of a deformed derivative opened new possibilities for, besides the thermodynamical, other treatment of complex systems, specially those with fractal or multifractal metrics and presenting long-range dynamical interactions. The deformation parameter or entropic index, q, occupying an important place in the description of those complex systems, describes deviations from standard Lie symmetries and the formalism aimed to accommodate scale invariance in a system with multifractal properties to the thermodynamic formalism. For $q \to 1$, the formalism reverts to the standard one.

Here, we consider the relevant space–time/ phase space as fractal or multifractal [15].

The use of deformed-operators is also justified here based on our proposition that there exists an intimate relationship between dissipation, coarse-grained media and the some limit scale of energy for the interactions. Since we are dealing with open systems, as commented in Ref. [1], the particles are indeed dressed particles or quasi-particles that exchange energy with other particles and the environment. Depending on the energy scale an interaction may change the geometry of space–time, disturbing it at the level of its topology. A system composed by particles and the surrounding environment may be considered nonconservative due to the possible energy exchange. This energy exchange may be the responsible for the resulting non-integer dimension of space–time, giving rise then to a coarse-grained medium. This is quite reasonable since, even standard field theory, comes across a granularity in the limit of Planck scale. So, some effective limit may also exist in such a way that it should be necessary to consider a coarse-grained space–time for the description of the dynamics for the system, in this scale. Also, another perspective that may be proposed is the previous existence of a nonstandard geometry, e.g., near a cosmological black hole or even in the space nearby a pair creation, that imposes a coarse-grained view to the dynamics of the open system. Here, we argue that deformed-derivatives, similarly to the FC, allows us to describe and emulate this kind of dynamics without explicit many-body, dissipation or geometrical terms in the dynamical governing equations. In some way, the formalism proposed here may yield an effective theory, with some statistical average without imposing any specific nonstandard statistics. So, deformed derivatives and/or FC may be the tools that could describe, in a softer way, connections between coarse-grained medium and dissipation at a certain energy scale.

One relevant applicability of our formalism may concern position-dependent systems (see Ref. [16] and references therein) that seems to be more adequate to describe the dynamics of many real complex systems where there could exist long-range interactions, long-time memories, anisotropy, certain symmetry breakdown, nonlinear media, etc. [17].

In the case of quantum mechanics, certain minimum length scale yields a modification in the position momentum commutation relationship or some modification in the underlying space–time that may results in a Schrödinger equation with a position-dependent mass [18].

Also, to find more suitable ways to explaining several complex behaviors in nature, Nonlinear (NL) equations have become an important subject of study [19], since the applicability of linear equations in physics is usually restricted to idealized systems [20].

An important point to emphasize is that the paradigm we adopt is different from the standard approach in the generalized statistical mechanics context, where the modification of entropy definition leads to the modification of algebra and consequently the derivative concept. We adopt that the mapping to a continuous fractal space leads naturally to the necessity of modifications in the derivatives, that we will call deformed or metric derivatives [21]. The modifications of derivatives brings to a change in the algebra involved, which in turn may conduct to a generalized statistical mechanics with some adequate definition of entropy.

In this paper, we initiate a general variational calculus with metric derivatives embedded into the Lagrangian. The purpose of the present work is to develop the corresponding generalization and for this, we define three options to pursue that will be described in the forthcoming sections.

Our paper is outlined as follows: In Section 2, we cast some mathematical aspects, in Section 3, we develop the variational approach based on embedded deformed-derivatives. In Section 4, we extend the formalism to relativistic independent fields. In Section 5, some applications are presented; in Section 6, the Hamiltonian formalism are addressed and we cast finally our Conclusions and Outlook in Section 7.