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Signals on graphs: Transforms and tomograms

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HIGHLIGHTS

- A general framework for graph signal transforms.
- A correct generalization of the notion of wavelet transform to graphs.
- The construction of the graph tomogram transforms.

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ABSTRACT

Development of efficient tools for the representation of large datasets is a precondition for the study of dynamics on networks. Generalizations of the Fourier transform on graphs have been constructed through projections on the eigenvectors of graph matrices. By exploring mappings of the spectrum of these matrices we show how to construct more general transforms, in particular wavelet-like transforms on graphs. For time-series, tomograms, a generalization of the Radon transforms to arbitrary pairs of non-commuting operators, are positive bilinear transforms with a rigorous probabilistic interpretation which provide a full characterization of the signals and are robust in the presence of noise. Here the notion of tomogram is also extended to signals on arbitrary graphs.

1. Introduction

The analysis of evolving data on complex networks is an emerging topic of current interest in physics and many other fields. Work in this field appears in several flavors. Some authors concern themselves with relations and connectedness, that is, with the community structure, others with the role of particular network vertices. A topic of relevance is the impact of network structure on the diffusion of information, imitation, failure propagation and all kinds of dynamical behaviors of the system. Reference models like random graphs, Poisson graphs, scale free graphs, etc. are regularly used to quantify the network characteristics, by using parameters which include clustering coefficients, path length, diameter and centralities. Inference and learning from large network-based datasets is also a topic of current interest.

Prior to the description of dynamical systems defined on graphs, or the dynamics of the graph itself, is the construction of efficient representations for these large datasets. Earlier works used spectral graph theory and the graph Laplacian [1] to derive low-dimensional representations by projecting the data on low-dimensional subspaces associated to subsets of the Laplacian eigenvectors. More recently some authors have proposed transforms for data indexed by graphs. In particular,

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generalizations of the Fourier transform have been proposed [2–4] which are used to extend to graphs many of the signal processing concepts used for time series [5–7]. Typically, these transforms make a change of basis from the vertex space to some other space of vectors which is then used to project the graph signal. They are the analog of linear transforms in time series. So far as we know, the question of multiple feature characterization of graph signals, for example bilinear transforms, has not been addressed in the past. For time series it is known that bilinear transforms, like Wigner–Ville, have serious interpretation problems. However the recently developed tomographic framework [8–11] allows for a probabilistically robust multiple feature characterization of time signals. The correct generalization of the notion of wavelet transforms to graphs and the extension of the tomographic framework to graph signals are the main contributions of this paper.

In Section 2, to set notation, we review the notions of linear, bilinear and tomographic transforms for time signals. In Section 3, transforms and tomograms for static and dynamical data on graphs are introduced and in Section 4 some illustrative examples are worked out.

2. Signal transforms for time series: Linear, quasi-distributions and tomograms

The traditional field of signal processing deals mostly with the analysis of time series. Signal processing of time series relies heavily on integral transforms [12,13]. Three types of transforms have been used: linear, bilinear and tomograms. Among the linear transforms, Fourier and wavelets are the most popular. The Fourier transform extracts the frequency components of the signal and the wavelets its multiscale nature. However, this is achieved at the expense of the time information, in the sense that the time location of the frequency components and of the scale features is lost in the process. This motivated the development of bilinear transforms like the time–frequency Wigner–Ville [14,15] or the frequency–scale Bertrand [16,17] quasidistributions. The aim of the Wigner–Ville transform was to provide joint information on the time–frequency plane, an important issue because, in many applications (biomedical, seismic, radar, etc.), the nature of the signals may change on short time scales. However, the oscillating cross-terms in the Wigner–Ville and other quasidistributions [18–20] render the interpretation of the transformed signals a difficult matter. Even when the average of the cross-terms is small, their amplitude may be large in time–frequency regions that carry no physical information.

The difficulties with the physical interpretation of quasidistributions arise from the fact that time and frequency (or frequency and scale) are noncommutative operator pairs. Hence, a joint probability density can never be defined. Even in the case of positive quasiprobabilities like the Husimi–Kano function [21,22], an interpretation as a joint probability distribution is also not possible because the two arguments in the function are not simultaneously measurable random variables. More recently, a new type of strictly positive bilinear transform has been proposed [8–10], called a tomogram, which is a generalization of the Radon transform [23] to arbitrary noncommutative pairs of operators. The Radon–Wigner transform [24,25] is a particular case of the noncommutative tomography technique. Being strictly positive densities, the tomograms provide a full characterization of the signal and are robust in the presence of noise.

A unified framework to characterize linear transforms, quasidistributions and tomograms was developed in Ref. [9]. In finite dimensional spaces a signal f may be represented as a column vector and the scalar product as $g^T f$, the transposed g^T being a row vector. However in infinite-dimensional spaces $\mathcal{N}$ and when f is not normalizable the notion of scalar product loses its meaning and is better to use the notation $|f\rangle$ and $\langle g|$ to emphasize that g belongs to a smaller space (the dual $\mathcal{N}^*$ of $\mathcal{N}$). Then, the notation $\langle g \mid f\rangle$, the value of the functional $\langle g|$ on the vector $|f\rangle$, generalizes the notion of scalar product when the space of the $\langle g|$'s and the $|f\rangle$'s cannot be identified. Also $\langle g| U |f\rangle$ means the action of the operator U on $|f\rangle$ and then the evaluation of the functional $\langle g|$ on the new vector [26].

Consider now a signal $f(t)$ as a vector $|f\rangle$ in a subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$, a family of unitary operators $U(\alpha) = e^{iB(\alpha)}$ and a reference vector h in the dual $\mathcal{N}^*$ of $\mathcal{N}$. A linear transform like Fourier or wavelet is

$$W_f^{(h)}(\alpha) = \langle U(\alpha) h \mid f\rangle \tag{1}$$

and a quasidistribution is

$$Q_f(\alpha) = \langle U(\alpha) f \mid f\rangle. \tag{2}$$

To define the tomogram let, in the unitary operator $U(\alpha) = e^{iB(\alpha)}$, $B(\alpha)$ have the spectral decomposition $B(\alpha) = \int XP(X)\,dX$, where $P(X)$ denotes the projector on the (generalized) eigenvector $\langle X| \in \mathcal{N}^*$ of $B(\alpha)$. The tomogram is

$$M_f^{(B)}(X) = \langle f| P(X) |f\rangle = |\langle X \mid f\rangle|^2 . \tag{3}$$

The tomogram $M_f^{(B)}(X)$ is the squared amplitude of the projection of the signal $|f\rangle \in \mathcal{N}$ on the eigenvector $\langle X| \in \mathcal{N}^*$ of the operator $B(\alpha)$. Therefore it is positive. For normalized $|f\rangle$,

$$\langle f \mid f\rangle = 1$$

the tomogram is normalized

$$\int M_f^{(B)}(X)\,dX = 1 \tag{4}$$