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Non-Euclidean-normed Statistical Mechanics

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HIGHLIGHTS

- Statistical Mechanics is generalized in the framework of non-Euclidean metrics induced by $\mathcal{L}^p$ norms.
- The non-Euclidean-normed canonical distribution for a power-law energy density states is formed.
- The range of possible values of the q-index, which depends on the value "p" of the $\mathcal{L}^p$-norm, is derived.
- The physical temperature coincides with the kinetically defined temperature.
- The new Statistical Mechanics follows the standard classical formalisms of thermodynamics.

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ABSTRACT

This analysis introduces a possible generalization of Statistical Mechanics within the framework of non-Euclidean metrics induced by the $\mathcal{L}^p$ norms. The internal energy is interpreted by the non-Euclidean $\mathcal{L}^p$-normed expectation value of a given energy spectrum. The presented non-Euclidean adaptation of Statistical Mechanics involves finding the stationary probability distribution in the Canonical Ensemble by maximizing the Boltzmann–Gibbs and Tsallis entropy under the constraint of internal energy. The derived non-Euclidean Canonical probability distributions are respectively given by an exponential, and by a q-deformed exponential, of a power-law dependence on energy states. The case of the continuous energy spectrum is thoroughly examined. The Canonical probability distribution is analytically calculated for a power-law density of energy. The relevant non-Euclidean-normed kappa distribution is also derived. This analysis exposes the possible values of the q- or κ-indices, which are strictly limited to certain ranges, depending on the given $\mathcal{L}^p$-norm. The equipartition of energy in each degree of freedom and the extensivity of the internal energy, are also shown. Surprisingly, the physical temperature coincides with the kinetically defined temperature, similar to the Euclidean case. Finally, the connection with thermodynamics arises through the well-known standard classical formalisms.

1. Introduction

The Euclidean $\mathcal{L}^2$-normed mean can be derived by minimizing the sum of the square ($\mathcal{L}^2$) deviations, that is the Euclidean variance. In particular, given the elements $\{y_i\}_{i=1}^N$, $y_i \in D_y \subseteq \Re, \forall i = 1, \ldots, N$, then the Total Square Deviations $TSD(\{y_i\}_{i=1}^N; \alpha)^2 = \sum_{i=1}^N |y_i - \alpha|^2$ are minimized for the optimal approximating finding value $\alpha^* = \mu_2$, that is the Euclidean mean, given by $\sum_{i=1}^N |y_i - \mu_2| \text{sign}(y_i - \mu_2) = 0$ ($\Leftrightarrow \mu_2 = \frac{1}{N}\sum_{i=1}^N y_i$). In a similar way, the non-Euclidean $\mathcal{L}^p$-normed means μ_p ($p \geq 1$) were deduced in Ref. [1], by minimizing the sum of the $\mathcal{L}^p$-normed deviations, or, Total p-Deviations,

$$TD_p(\{y_i\}_{i=1}^N; \alpha; p)^p = \sum_{i=1}^N |y_i - \alpha|^p. \tag{1}$$

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The optimization leads to the normal equation

$$\sum_{i=1}^{N} |y_i - \mu_p|^{p-1} \mathrm{sign}(y_i - \mu_p) = 0, \tag{2}$$

from which the optimal parameter $\alpha^* = \mu_p$, that is the non-Euclidean $\mathcal{L}^p$-normed mean, is derived as an implicit expression of p. (See also: Refs. [2,3].)

This generalization obeys in a formal scheme of means characterization, given by the univalued, N-multivariable function $M(\{y_i\}_{i=1}^N)$, fulfilling the three preconditions: (i) Continuity; (ii) Internness: $\mathrm{Min}(\{y_i\}_{i=1}^N) \le M(\{y_i\}_{i=1}^N) \le \mathrm{Max}(\{y_i\}_{i=1}^N)$; (iii) Symmetry: For $y_i \to y_{j_i}$, $\forall\, i = 1, \ldots, N$: $\{y_i\}_{i=1}^N = \{y_{j_i}\}_{i=1}^N$, then $M(\{y_i\}_{i=1}^N) = M(\{y_{j_i}\}_{i=1}^N)$.

Furthermore, given the spectrum of y-values $\{y_k\}_{k=1}^W$, associated with the possibilities $\{p_k\}_{k=1}^W$, the concept of expectation value is generalized to the non-Euclidean adaptation $\langle y\rangle_p$ that is implicitly expressed by

$$\sum_{k=1}^{W} p_k |y_k - \mu_p|^{p-1} \mathrm{sign}(y_k - \mu_p) = 0, \tag{3}$$

where the classical Euclidean expectation value is recovered for $p = 2$, i.e., $\langle y\rangle_2 = \sum_{k=1}^W p_k y_k$. According to this, the internal energy U_p of a system that characterizes by a discrete energy spectrum $\{\varepsilon_k\}_{k=1}^W$, associated with the discrete probability distribution $\{p_k\}_{k=1}^W$, is interpreted by the non-Euclidean $\mathcal{L}^p$-normed expectation value, that is implicitly given by

$$\sum_{k=1}^{W} p_k |\varepsilon_k - U_p|^{p-1} \mathrm{sign}(\varepsilon_k - U_p) = 0. \tag{4}$$

This is written in terms of the non-Euclidean norm operator $\hat{\mathcal{L}}_p$, defined by

$$\hat{\mathcal{L}}_p(\varepsilon_k) = \frac{|\varepsilon_k - U_p|^{p-1} \mathrm{sign}(\varepsilon_k - U_p)}{(p-1)\phi_p} + U_p, \tag{5}$$

namely,

$$U_p = \langle \hat{\mathcal{L}}_p(\varepsilon)\rangle_2 = \sum_{k=1}^{W} p_k \hat{\mathcal{L}}_p(\varepsilon_k), \quad \text{or}, \quad \langle \hat{\mathcal{L}}_p(\varepsilon - U_p)\rangle_2 = 0. \tag{6}$$

The argument ϕ_p is defined by

$$\phi_p \equiv \sum_{k=1}^{W} p_k |\varepsilon_k - U_p|^{p-2}, \tag{7}$$

that is the appropriate expression for deducing the zero-mean of the derivative of $\{\hat{\mathcal{L}}_p(\varepsilon_k)\}_{k=1}^W$ with respect to a given parameter β, for which $p_k = p_k(\{\varepsilon_{k'}\}_{k'=1}^W; \beta)$, $\forall\, k = 1, \ldots, W$, and $U_p = U_p(\{\varepsilon_k\}_{k=1}^W; \beta)$, i.e.,

$$0 = \sum_{k=1}^{W} p_k \frac{\partial}{\partial \beta} \hat{\mathcal{L}}_p(\varepsilon_k), \quad \text{or}, \quad \frac{\partial}{\partial \beta}\langle \varepsilon\rangle_p = \sum_{k=1}^{W} \frac{\partial p_k}{\partial \beta} \hat{\mathcal{L}}_p(\varepsilon_k). \tag{8}$$

In Ref. [1] the argument ϕ_p was the key-point for extracting the exact expression of the $\mathcal{L}^p$-normed variance. In addition, the zero-mean property, given in Eq. (8), is fulfilled if and only if the argument ϕ_p has the specific expression given in Eq. (7). Only then, the Canonical probability distribution can be derived. If ϕ_p were expressed by any other formulation, after the extremization of entropy in the Canonical Ensemble, we would not be able to solve in terms of the probability. Moreover, the zero-mean property helps to connect the non-Euclidean Statistical Mechanics with Thermodynamics. (Regarding the expression of ϕ_p and the importance of the zero-mean property, see Ref. [1].)

As another point of view, [4] generalized the ordinary (Euclidean) expectation value $\langle\varepsilon\rangle_2 = \sum_{k=1}^W p_k \varepsilon_k$ to the escort expectation value, i.e.,

$$\langle \varepsilon\rangle_q = \sum_{k=1}^{W} P_k \varepsilon_k, \tag{9}$$

where the escort probability distribution $\{P_k\}_{k=1}^W$ is constructed via the ordinary probability distribution $\{p_k\}_{k=1}^W$ and the duality relation [5–7]

$$P_k \equiv {p_k}^q \Big/ \sum_{k'=1}^{W} {p_{k'}}^q, \qquad p_k = {P_k}^{1/q} \Big/ \sum_{k'=1}^{W} {P_{k'}}^{1/q}. \tag{10}$$