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Semi-directed percolation in two dimensions

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HIGHLIGHTS

- Semi-directed percolation model on the square and triangular lattices is studied.
- Transfer-matrix and phenomenological renormalization approaches are used.
- Good estimates of critical exponents and critical probabilities are obtained.
- These results provide numerical evidence that semi-directed percolation belongs to the universality class of fully-directed percolation.

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ABSTRACT

We studied a model of semi-directed percolation on finite strips of the square and triangular lattices. Using the transfer-matrix method, combined with phenomenological renormalization group approach, we obtain good numerical estimates for critical probabilities and correlation lengths critical exponents. Our results confirm the conjecture that semi-directed percolation belongs to the universality class of the usual fully-directed percolation model.

1. Introduction

The percolation problem on lattices is the subject of a lasting interest in statistical physics [1], but also in the probability theory [2]. It is well known that connectivity properties of percolation structure change at percolation threshold, and that this transition can be described in terms of the usual concepts of critical phenomena. It is also known that the introduction of a preferred connectivity direction in the percolation problem leads to an anisotropic scaling, which opens a new class of universality — the directed percolation critical behavior. Interest for directed percolation (DP) has been further enhanced by the fact that critical dynamics of many other simple processes belongs to this, rather robust, universality class [3].

While DP has been much studied in the past, the semi-directed percolation (SDP) problem, which can be seen as an intermediary model between the DP and the usual isotropic percolation model (see Fig. 1), has not received a comparable attention. It appears to us that semi-directed bond percolation on the square lattice has been examined for the first time in statistical physics through a more general model — random resistor–diode networks [4]. Several rigorous results, concerning bounds on the values of semi-directed percolation thresholds, have appeared in mathematical literature (see e.g. Ref. [5] and references therein). Various studies of closely related models of semi-directed lattice animals [6–10] indicated that they presumably belongs to the universality class of fully-directed animals. Let us also note that exact results obtained

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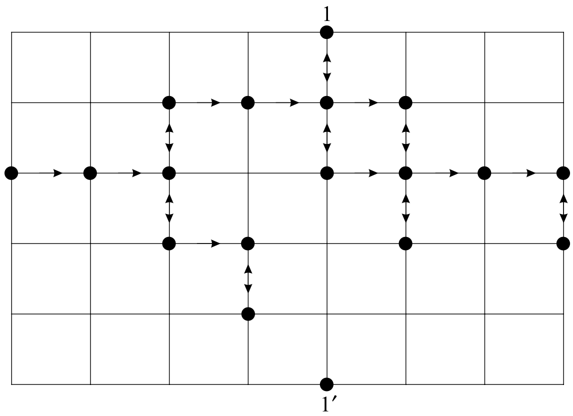

Fig. 1. A semi-directed site percolation cluster on the square lattice, containing $N = 17$ sites (filled circles) on a strip of width $n = 5$; due to periodic boundary conditions the sites denoted by 1 and 1′ should be identified. The strip is infinite along the horizontal axis, which also represents the preferred direction. In contrast to the case of fully-directed percolation, vertical lines of the lattice are not directed, which means that the set of all semi-directed percolation clusters includes the set of all possible fully-directed clusters.

in studies of semi-directed self-avoiding walk models [9,11,12] show that they indeed have the same critical properties as corresponding fully-directed ones. There is no, however, analogous exact results in the case of percolation, and mathematical proof that SDP and DP have the same critical behavior seems to be difficult. Besides, an early study of SDP on finite strips of two-dimensional lattices [13] revealed rather poor convergence of effective percolation threshold probabilities to their limiting values for infinite system. In the absence of reliable estimates of critical exponents for SDP model, the authors of Ref. [13] in their extrapolation approach used the best available values of critical exponents for DP model to improve convergence of critical probabilities in the former model.

The aim of our paper is to obtain as precise as possible estimates for both critical probabilities and critical exponents of SDP model, without referring to the DP model. To achieve this goal we extended the previous study of SDP on finite strips by doubling their sizes, and we used a different extrapolation procedure to get better estimates of quantities of interest on the square and triangular lattice. In this way, we have been able to get accurate estimates of correlation length critical exponents, for the first time, and to improve significantly the precision of estimates for critical probabilities. The obtained results support the conjecture that SDP and DP model share the same universality class.

In Section 2 we describe the SDP model and outline our computational approach. Our numerical results and extrapolation procedures are described in Section 3. In Section 4 we give a summary of the obtained results.

2. Model and approach

In this paper we consider critical properties of semi-directed site percolation clusters on the square and triangular lattice (Fig. 1 represents an illustration for the square lattice). As usual, we assume that each lattice site is present with probability p and that it is absent with probability $1 - p$. Main quantity of interest in the transfer-matrix approach to the percolation problem on strips is the probability $\mathcal{P}_L(p)$ that column L is connected to the first column of the strip [14]. It has been shown [14] that this quantity can be expressed in terms of a finite number of restricted probabilities $P_L(\mathcal{C}_k)$, where $\mathcal{C}_k$, $k = 1, 2, \ldots \mathcal{N}$ denotes one of $\mathcal{N} = \mathcal{N}(n)$ different site configurations at column L. One can directly relate probabilities $P_{L+1}(\mathcal{C}_k)$ to $P_L(\mathcal{C}_i)$ via $\mathcal{N}$ linear recursion relations, and thus to define a transfer-matrix **M** which is independent of L,

$$P_{L+1}(\mathcal{C}_k) = \sum_{i=1}^{\mathcal{N}} \mathbf{M}_{ki}(p) P_L(\mathcal{C}_i). \tag{1}$$

For large L all these probabilities decay in general exponentially: $P_L(\mathcal{C}_k) \sim \exp(-L/\xi_n)$, with ξ_n being a characteristic (correlation) length.

It is expected that in the case of directed percolation models there are two different characteristic lengths: one of them provides a measure of average cluster longitudinal size $\xi^{\parallel}$, while the other $\xi^{\perp}$ characterizes cluster size measured along the perpendicular direction. Near the percolation threshold p_c these two lengths follow the power law: $\xi^{\parallel} \sim (p - p_c)^{-\nu_{\parallel}}$ and $\xi^{\perp} \sim (p - p_c)^{-\nu_{\perp}}$, where $\nu_{\parallel}$ and $\nu_{\perp}$ are two different critical exponents.

According to finite-size scaling theory for directed systems, the correlation lengths $\xi_n^{\parallel}$ calculated on finite strips of width n scales on the following way

$$\xi_n^{\parallel} = n^{\nu_{\parallel}/\nu_{\perp}} F[n^{1/\nu_{\perp}}(p - p_c)], \tag{2}$$