Physica A xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

# A family of models for Schelling binary choices

Fausto Cavalli [a,1], Ahmad Naimzada [a,2], Marina Pireddu [b,*]

[a] Department of Economics, Management and Statistics, University of Milano-Bicocca, U6 Building, Piazza dell'Ateneo Nuovo 1, 20126 Milano, Italy

[b] Department of Mathematics and Applications, University of Milano-Bicocca, U5 Building, Via Cozzi 55, 20125 Milano, Italy

## HIGHLIGHTS

- We introduce a family of discrete dynamical systems to model Schelling binary choices.
- We study steady states and their relation to the equilibria predicted by Schelling.
- Local stability, possible destabilizations and chaos existence are analyzed.
- Using bifurcation theory, we study scenarios qualitatively described by Schelling.
- We provide examples and simulations that confirm the analytical results.

## ARTICLE INFO

*Article history:*
Received 21 January 2015
Received in revised form 9 September 2015
Available online xxxx

*Keywords:*
Schelling binary choices
Local stability
Bifurcations
Chaotic dynamics

## ABSTRACT

We introduce and study a family of discrete-time dynamical systems to model binary choices based on the framework proposed by Schelling in 1973. The model we propose uses a gradient-like adjustment mechanism by means of a family of smooth maps and allows understanding and analytically studying the phenomena qualitatively described by Schelling. In particular, we investigate existence of steady states and their relation to the equilibria of the static model studied by Schelling, and we analyze local stability, linking several examples and considerations provided by Schelling with bifurcation theory. We provide examples to confirm the theoretical results and to numerically investigate the possible destabilizations, as well as the emergence of coexisting attractors. We show the existence of chaos for a particular example.



## 1. Introduction

A binary choice is an either-or situation, in which agents have exactly two possible choices, which we will indicate in what follows by $R$ and $L$. It is a very simple setting which, however, allows describing a wide variety of different situations, ranging from the decision to drive or not on certain days, to take or not an antibiotic, to take part in a collective action or to stay inactive and so on. The criteria according to which an agent makes a decision or its opposite may be different and related to both the decision itself and its consequences, and the influence exerted by the choices of the other agents on his own choice. In particular, in the latter case, we have that the choice of each individual can be influenced by social interactions. For instance, in the context of statistical physics, models with local interactions may be found in Refs. [1,2], while in Ref. [3] the authors study the binary choices of a set of agents through mean-field analytical methods.

* Corresponding author. Tel.: +39 0264485767; fax: +39 0264485705.
*E-mail addresses:* fausto.cavalli@unimib.it (F. Cavalli), ahmad.naimzada@unimib.it (A. Naimzada), marina.pireddu@unimib.it (M. Pireddu).
1 Tel.: +39 0264485879; fax: +39 0264483085.
2 Tel.: +39 0264485813; fax: +39 0264483085.

http://dx.doi.org/10.1016/j.physa.2015.10.038
0378-4371/© 2015 Published by Elsevier B.V.

A fundamental contribution about the effects of social interactions on binary choices is the seminal work by Schelling in Ref. [4], in which a systematic discussion of several binary choice contexts is presented, together with their qualitative study. A key assumption in Ref. [4] is that agents are supposed to be *rational*. The presence of rational agents is a fundamental aspect, and rational choice models are an important class of mathematical sociology models (see Coleman [5], Sorensen and Sorensen [6], and, more recently, Braun [7]). The work by Schelling in Ref. [4] provided a general framework suitable for the development of a wide range of models and in fact it has been taken into account in several works during the past decades. Without claiming to be complete, we can mention the influence of Ref. [4] on modeling resource management (Runge [8]), corruption diffusion (Andvig and Moene [9]), public opinion formation (Dodds and Watts [10]) and also nature sciences, as well as the diffusion of contagion (Dodds and Watts [11]).

Going back to the rationality assumption on agents, it allows studying possible equilibria in terms of the payoff functions associated to the individual choices. The payoff functions are assumed to depend only on the number (and not on the identity) of the agents that make a particular choice, so that those functions can be assumed to depend just on the fraction $x$ of agents making, for instance, choice $R$. The agents opt for a choice or its opposite on the basis of the payoff. However, since, for each agent, the choice of the others influences his own payoff, he has to take into account also the effect of the others choice. This is called *externality*, and in Ref. [4] it is also taken into account for the classification of the different binary choices with respect to related payoffs. In the qualitative model by Schelling, *equilibria* correspond to the Nash equilibria and occur

- at the internal fractions of agents for which the payoff functions intersect (or, equivalently, the payoff differential is null) and no agents take advantage from changing its choice;
- in some extremal situations, in which all agents are making the same decision, provided that the payoff of such choice is larger than that of the opposite one.

For example, Schelling formalizes the multiperson prisoner's dilemma (MPD), for which the following conditions have to be satisfied:

1. each individual faces the same binary choice and payoffs;
2. each individual has a dominant choice independently of what the others do;
3. any individual is better off the more numerous are the agents who make their dominated choice.

In this situation, the payoff functions do not intersect and the equilibrium is represented by the configuration in which each agent chooses the dominated strategy. In this case, the externality is uniform, in the sense that when the fraction of agents making a particular choice is increased, it has the same effects on the payoffs of agents choosing both $R$ and $L$. For example, imagine that both payoffs are described, as in the MPD, by increasing functions. If an agent chose $R$, then the payoffs of the other agents, who chose both $R$ and $L$, would increase, while if an agent chose $L$, both $R$ and $L$ payoffs would decrease. Then, in the first case we speak of positive externality, while in the second one of negative externality.

Another framework that can be classified by means of the concept of externality is that of common goods. A concrete example of such situation is described by Schelling when he examines the decision to use or not a car with respect to the issue of traffic congestion. In this case, the two payoff functions have opposite monotonicity, intersect once and we have an internal stable equilibrium, different from the extremal situations in which all agents make the same decision. In such context, externality is contingent, as the effects of a given choice are not the same on both payoffs. For example, if the payoff of choice $R$ is decreasing and that of choice $L$ is increasing, choosing $R$ increases the payoff of $L$ and decreases that of $R$, while choosing $L$ has an opposite effect.

A further example concerning externalities is given by the so-called network goods, in which the users of a good gain when additional users adopt it. This is somehow the opposite of the common goods example, since now the internal intersection for the payoff functions is no more a stable steady state, but it rather acts as a discriminating level between the extremal states in which all players make the same choice.

Such examples show that externalities are essentially connected with the monotonicity of the payoff functions. However, the monotonicity of the payoff functions alone is not sufficient to catch all the possible dynamic behaviors originating with binary choices. Schelling notices, for instance, that also the steepness of the payoff functions is relevant: "*… classification has to consider … whether or not the externality favors more the choice that yields the externality. That is, with a Right choice yielding the positive externality, does it yield a greater externality to a Right choice or to the Left? Which curve is steeper?*" [4, p. 403].

In Ref. [4], it is also marginally considered the problem of how the situation changes when one of the payoff functions is kept fixed and the other is changed, for example rotated (see Ref. [4, p. 404]). Clearly, the static qualitative analysis of Ref. [4] does not allow to study what happens to the fractions dynamics when the payoff curves are varied, for example depending on a parameter that regulates their slope or position.

The aim of this work consists in providing a modeling framework based on a discrete dynamic model in order to analytically study the dynamics underlying the qualitative and essentially static setting in Schelling [4]. Taking into account a time-dependent model, we want to provide an explicit dynamic adjustment mechanism to validate and deepen the analysis by Schelling, focusing on the local properties of the equilibria, on their stability and on the causes of stability loss, together with the possible scenarios arising when equilibria become unstable. In our contribution we focus on social and economic issues that can be modeled by means of discrete-time processes, i.e., processes in which decisions do not change continuously in time, but rather require some time to be modified (for related examples, see for instance Ref. [12]). We also



Please cite this article in press as: F. Cavalli, et al., A family of models for Schelling binary choices, Physica A (2015), http://dx.doi.org/10.1016/j.physa.2015.10.038