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Structural stability of Lattice Boltzmann schemes

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HIGHLIGHTS

- Our goal is to determine classes of traveling solitary wave solutions for Lattice Boltzmann schemes by means of a hyperbolic ansatz.
- Such lattice solitary waves are not solutions of the exact continuous original equation.
- The occurrence of such spurious solitary waves might cause structural instability of the scheme.
- The existence of spurious numerical lattice solitary waves for Lattice Boltzmann schemes is eventually proved.

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ABSTRACT

The goal of this work is to determine classes of traveling solitary wave solutions for Lattice Boltzmann schemes by means of a hyperbolic ansatz. It is shown that spurious solitary waves can occur in finite-difference solutions of nonlinear wave equation. The occurrence of such a spurious solitary wave, which exhibits a very long life time, results in a non-vanishing numerical error for arbitrary time in unbounded numerical domain. Such a behavior is referred here to have a structural instability of the scheme, since the space of solutions spanned by the numerical scheme encompasses types of solutions (solitary waves in the present case) that are not solutions of the original continuous equations. This paper extends our previous work about classical schemes to Lattice Boltzmann schemes (David and Sagaut 2011; 2009a,b; David et al. 2007).

1. Introduction

The lattice Boltzmann method (*LBM*) is used for the numerical simulation of physical phenomena, and serves as an alternative to classical solvers of partial differential equations. The primary domain of application is fluid dynamics; it is specially used to obtain the numerical solution of the incompressible, time-dependent Navier–Stokes equation.

The strength of the Lattice Boltzmann method is due to its ability to easily represent complex physical phenomena, ranging from multiphase flows to fluids with chemical reactions. The principle is to "mimic" at a discrete level the dynamics of the Boltzmann equation. Since it is based on a molecular description of a fluid, the knowledge of the microscopic physics can directly be used to formulate the best fitted numerical model.

This method can be regarded as either an extension of the lattice gas automaton (*LGA*) [1–3], or a special discrete form of the Boltzmann equation from kinetic theory. Although the connection between the gas kinetic theory and hydrodynamics has long been established, the Lattice Boltzmann method (*LBM*) needs additional special discretization of velocity space to

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recover the correct hydrodynamics. Due to the very same reason, the *LBM* works exactly opposite traditional *CFD* methods in deriving working schemes: *LBM* uses Navier–Stokes equations as its target while traditional *CFD* methods use Navier–Stokes equations as their starting point.

Until a few years, the *LBM* was applicable to the isothermal flow regime, i.e., the weakly compressible, low-Mach-number limit. This flow regime is traditionally treated as "incompressible", although there are *CFD* methods constructed to compute the Navier–Stokes equations in this regime. The argument for treating very low-Mach-number flows as incompressible is pragmatic rather than physical. The direct calculation of the isothermal Navier–Stokes equations requires time steps sufficiently small to resolve acoustic waves across a computational cell. This time step may be vastly smaller than the time scales of interest for the bulk fluid motion. Thus the computational cost of the many additional time steps required by an isothermal calculation may be vastly higher than the cost of an incompressible calculation. Of course, in reality there is no fluid or flow that is absolutely incompressible (i.e., with infinite acoustic velocity). Recent works have shown that it is possible to define lattices able to overcome this limitation (see, for instance, Ref. [4], where the authors lay the theoretical foundation of the lattice Boltzmann model for the simulation of flows with shock waves and contact discontinuities, also Ref. [5], where one can find a powerful scheme the computational convergence rate of which can be improved compared to classical previous ones, while proving to be efficient for Taylor vortex flow, Couette flow, Riemann problem).

As for the enhancement of stability, which used to be pointed out in most lattice Boltzmann models, the regularization technique has been successfully shown as an efficient method of stabilization of the Single-Relaxation-Time (SRTLB) (see Ref. [6], where the model proposed by the authors enables one to raise significantly the maximum Reynolds number that could be simulated at a given level of grid resolution, in two and three dimensions).

2. The lattice Boltzmann method

The lattice Boltzmann (*LB*) method follows the same idea as its predecessor the Lattice Gas Automata (*LGA*) when it also considers the fluid on a lattice with space and time discrete. Instead of directly describing the fluid by discrete particles and, thus Boolean variables, it describes the fictitious system in terms of the probabilities of presence of the fluid particles. A lattice Boltzmann numerical model simulates the time and space evolution of kinetic quantities, the particle distribution functions $f_j(\vec{r}, t), 0 \leqslant j \leqslant J, J \in \mathbb{N}^\star$.
The lattice Boltzmann equation is obtained by ensemble averaging the equation

$$\left\langle N_j\left(\vec{r} + \Delta t\, \vec{v}_j, t + \Delta t\right)\right\rangle = \left\langle N_j\left(\vec{r}, t\right)\right\rangle + \left\langle \Omega_j\left(N\right)\right\rangle \tag{1}$$

where $\left\langle N_j\left(\vec{r}, t\right)\right\rangle$ denotes the average number of particles at space position $\vec{r}$ and time t.
The system is supposed to satisfy the Boltzmann molecular chaos hypothesis, i.e. the fact that there is no correlation between particles entering a collision. Thus, the collision operator can be expressed as $\langle \Omega_i\left(N\right)\rangle = \Omega_i\langle N\rangle$, which leads to the *Lattice Boltzmann equation*:

$$f_j\left(r + \Delta t\, v_j, t + \Delta t\right) = f_j\left(r, t\right) + \Omega_j\left(f\right) \tag{2}$$

where, for $j \in \mathbb{N}, f_j = \langle N_j\rangle$ denotes the probability to have a fictitious fluid particle of velocity v_j entering lattice site $\vec{r}$ at time t. The f_j are also called the *fluid fields*, or the *particle distribution functions*.

The collision operator is normally a non-linear expression and requires a lot of computation time [7]. In a big lattice, e.g. 3D model, the computation becomes impossible even on a massively parallel computer. To overcome this problem, Higuera et al. [8,9] proposed to linearize the collision operator around its local equilibrium solution to reduce the complexity of the operation. Using this idea, Bhatnagar, Gross and Krook introduced the *BGK lattice* (*LBGK*) [10], in which the collision between particles is described in terms of the relaxation towards a local equilibrium distribution. The *LBGK* is considered to be one of the simplest forms of the Lattice Boltzmann equation and is mathematically expressed as

$$f_j(\vec{r} + \vec{e}_j\, t + \Delta t) = f_j(\vec{r}, t) - \frac{1}{\tau}\left\{f_j(\vec{r}, t) - f_j^{eq}(\vec{r}, t)\right\} \quad j \in \{0, \ldots, J\}, J \in \mathbb{N}^\star \tag{3}$$

where τ is the single relaxation time, which is a free parameter of the model to determine the fluid viscosity, and f_j^{eq}, $j0, \ldots, J$, denote the local equilibrium functions, which are functions of the density and the flow velocity $\vec{u}$.
In the lattice Boltzmann method, the space variable vector $\vec{r}$ is supposed to live in a lattice $\mathcal{L}$ included in an Euclidian space of dimension d, $d \geqslant 1$.
The velocity belongs to a finite set $\mathcal{V}$ composed by given velocities $\vec{e}_j, j \in \{0, \ldots, J\}, J \in \mathbb{N}^\star$, chosen in such a way that

$$\vec{r} \in \mathcal{L} \text{ and } \vec{e}_j \in \mathcal{V} \Rightarrow \vec{r} + \Delta t\, \vec{e}_j \in \mathcal{L} \tag{4}$$

where Δt denotes the time step.
The set of velocities $\mathcal{V}$ is supposed to be invariant by space reflection, i.e.:

$$\vec{e}_j \in \mathcal{V} \Rightarrow \exists\, l \in \mathcal{V} : \vec{e}_l = -\vec{e}_j \in \mathcal{V}. \tag{5}$$

The numerical scheme is thus defined through the evolution of the population $f_j(\vec{r}, t)$, with $\vec{r} \in \mathcal{L}$ and $j \in \{0, \ldots, J\}$ towards a distribution $f_j(\vec{r}, t + \Delta t)$ at a new discrete time.