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Pseudo-ϵ expansion and critical exponents of superfluid helium

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HIGHLIGHTS

- Pseudo-ϵ expansions for critical exponents of λ-transition in ^{4}He are presented up to six-loop order.
- Numerical estimates extracted from the pseudo-ϵ expansions agree quite well with the experimental data.
- Pseudo-ϵ expansion is a powerful tool robust enough to evaluate critical exponents with very small errors.

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ABSTRACT

The pseudo-ϵ expansions (τ-series) for critical exponents of the three-dimensional *XY* model describing λ-transition in liquid helium are derived up to the τ^6 terms. Numerical estimates extracted from the τ-series obtained using the Padé–Borel resummation technique, scaling relations and the seven-loop (τ^7) estimate for the Fisher exponent η are presented including those for the exponents α and ν measured in experiments with a record accuracy. For the exponent α the procedure argued to be most reliable gives $\alpha = -0.0117$, the number that is very close to the most accurate experimental values. It signals that the pseudo-ϵ expansion approach is a powerful tool robust enough to evaluate critical exponents with small absolute errors. The arguments in favor of such a robustness are presented.

1. Introduction

Nowadays there exists a great number of high-precision numerical estimates of critical exponents and of other universal quantities obtained within various theoretical approaches. Use of high-temperature expansions, Monte Carlo simulations, the field-theoretical renormalization-group analysis based upon many-loop calculations in three and $(4 - \epsilon)$ dimensions resulted in theoretical estimates having (apparent) accuracy comparable or higher than that of the experimental data (see, e.g. Refs. [1–3]). Among the field-theoretical approaches, the method of pseudo-ϵ expansion suggested by B. Nickel (see Ref. 19 in the paper [4]) is of particular interest. This method was applied to variety of systems [4–12] including two-dimensional ones and those with non-trivial symmetry of the order parameter and lead to rather good numerical results for all the models studied. High numerical power of the pseudo-ϵ expansion technique stems from its key feature: it transforms strongly divergent renormalization group (RG) expansions into the series having smaller lower-order coefficients and much slower growing higher-order ones what makes them very convenient for getting numerical estimates. The pseudo-ϵ expansion

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machinery works well even in the case of the Fisher exponent η when the original RG expansion has irregular structure and is quite unsuitable for extracting numerical results [13].

In this paper, we apply the method of pseudo-ϵ expansion to the system for which the critical exponents were measured with an unprecedented accuracy, i.e. to the superfluid helium-4. Traditionally [14,15], the experimental study of thermodynamic and kinetic properties of this quantum fluid is carried out on the very high technical level and covers the temperatures extremely close to the λ point. This enabled one to determine the critical exponents with extremely small uncertainty, including the specific heat exponent α which is known to be tiny. The record measurements [16] performed in space in order to avoid the influence of gravity yielded $\alpha = -0.0127 \pm 0.0003$, the value of critical exponent that is accepted as the most accurate ever obtained experimentally. Below the pseudo-ϵ expansions (τ-series) for critical exponents of 3D *XY* model describing the critical behavior of the superfluid helium-4 will be calculated starting from the six-loop [17] RG series. The τ-series for the exponents α and γ will be written down up to the τ^6 terms. Numerical estimates for the critical exponents will be obtained using the Padé–Borel resummation technique, the scaling relations and the seven-loop (τ^7) pseudo-ϵ expansion estimate for the Fisher exponent η. Comparing the numbers obtained with the results of the most advanced experiments and with the values extracted from the lattice and field-theoretical calculations a numerical effectiveness of the pseudo-ϵ expansion approach will be evaluated. The general properties of this approach will be discussed and the roots of its high numerical power will be cleared up.

2. Pseudo-ϵ expansions for critical exponents and numerical estimates

The critical thermodynamics of the 3D *XY* model is described by Euclidean field theory with the Hamiltonian:

$$H = \int d^3x \left[\frac{1}{2}(m_0^2 \varphi_\alpha^2 + (\nabla \varphi_\alpha)^2) + \frac{\lambda}{24}(\varphi_\alpha^2)^2 \right], \tag{1}$$

where $\alpha = 1, 2$, $m_0^2 \sim T - T_c^{(0)}$, $T_c^{(0)}$ being the mean field transition temperature. We derive the pseudo-ϵ expansions for critical exponents α and γ starting from the known six-loop RG series [17]. To find the pseudo-ϵ expansions one has to substitute the τ-series for the Wilson fixed point coordinate g^* into RG expansions for α and γ and reexpand them in τ. With the τ-series for g^* [12] and the RG series for the critical exponents in hand the calculations are straightforward. Their results read:

$$\alpha = \frac{1}{2} - \frac{3\tau}{10} - 0.1297777778\tau^2 - 0.039547352\tau^3 - 0.02432025\tau^4 - 0.00324983\tau^5 - 0.0121092\tau^6. \tag{2}$$

$$\gamma^{-1} = 1 - \frac{\tau}{5} - 0.0405925926\tau^2 + 0.004326858\tau^3 - 0.00566467\tau^4 + 0.00458218\tau^5 - 0.0067372\tau^6. \tag{3}$$

We do not present here the τ-series for the critical exponent ν and others since they can be easily deduced from (2), (3) using well known scaling relations.

Despite small and rapidly decreasing coefficients the pseudo-ϵ expansions (2), (3) remain divergent. So, to extract the numerical values of the critical exponents from these series one has to apply some resummation procedure. We employ the Padé–Borel resummation technique based upon the Borel transformation

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = \int_0^\infty e^{-t} F(xt)\, dt, \qquad F(y) = \sum_{i=0}^{\infty} \frac{c_i}{i!} y^i \tag{4}$$

and use the Padé approximants [L/M] for an analytical continuation of the Borel transform $F(y)$. An application of this technique to the τ-series for α leads, however, to the results which are far from to be satisfactory. This is seen from Table 1 representing the Padé–Borel triangle for the series (2). More than a half of nontrivial estimates are absent in this table because of the positive axis ("dangerous") poles spoiling corresponding Padé approximants. Existing estimates are strongly scattered being practically useless for getting accurate value of the exponent α. Moreover, even more conservative procedure that uses simple Padé approximants and gives numerical results much less sensitive to the problem of poles results in the numbers appreciably differing from each other even in the highest (τ^6) order available. Table 2 representing the Padé triangle for the series (2) demonstrates this fact.

In such a situation it is natural to evaluate the exponent α in a different manner, using the scaling relations containing the critical exponents ν, γ and η:

$$\alpha = 2 - D\nu, \qquad \nu = \frac{\gamma}{2 - \eta}. \tag{5}$$

This way to evaluate α looks attractive because of two reasons. First, the Padé–Borel estimates of γ resulting from the τ-series (3) converge to the asymptotic value very rapidly signaling that for this exponent the iteration procedure employed is rather efficient. This is clearly seen from Table 3 where the Padé–Borel triangle for the exponent γ is presented. Second, the numerical value of the Fisher exponent can be extracted from the recently found seven-loop τ-series [13], i.e. it can be obtained with the highest accuracy accessible within the pseudo-ϵ expansion approach.