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A note on the fractional logistic equation

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HIGHLIGHTS

- Exact solution recently proposed to the fractional logistic equation is discussed.
- Properties of Mittag-Leffler functions are used in our reasoning.
- Differences between Mittag-Leffler functions and exponential function are pointed out.

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ABSTRACT

In this short note, we show that the real function recently proposed by West (2015) is not an exact solution for the fractional logistic equation.

1. Introduction

The topic of Fractional Calculus (that is, the study of integrals and derivatives of non-integer order) is nowadays a growing interest area of mathematics. Moreover, physicists and engineers are also really interested in applications coming from this nice theory which origin goes back to the theory of differential calculus; more precisely to Leibniz's famous note in his letter to L'Hôpital dated 30 September 1695.

For a long time, the theory of Fractional Calculus developed only as a theoretical field of mathematics. However, in the last decades, it was shown that some fractional operators describe in a better way some complex physical phenomena, especially when dealing with memory processes or viscoelastic and viscoplastic materials. Well known references about the application of fractional operators in rheology modeling are Refs. [1,2]. One of the most important advantage of fractional order models in comparison with integer order ones is that fractional integrals and derivatives are a powerful tool for the description of memory and hereditary properties of some materials. Notice that integer order derivatives are local operators, but the fractional order derivative of a function in a point depends on the past values of such function. This features motivated the successful use of fractional calculus in CRONE [3] and PID controllers [4, Chapter 9].

Due to this, in the last two or three decades, a great interest has been devoted to the study of fractional differential equations. Thus, fractional order differential equations (one day called extraordinary differential equations) play today a

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very important role describing some real world phenomena. For a complete exposition of the theory of Fractional Calculus one can see Refs. [5,6,4,7].

The exponential function, $\exp(t)$, plays a fundamental role in mathematics and it is really useful in the theory of integer order differential equations. In the case of fractional order, it loses some beautiful properties and Mittag-Leffler function appears as its natural substitute. Next, we recall its definition and some basic properties.

Definition 1.1. The function $E_\alpha(z)$ is named after the great Swedish mathematician Gösta Mittag-Leffler (1846–1927) who defined it as a power series given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

This function provides a simple generalization of the exponential function because of the replacement of $k! = \Gamma(k+1)$ by $(\alpha k)! = \Gamma(\alpha k + 1)$ in the denominator of the terms of the exponential series. Due to this, such function can be considered the simplest nontrivial generalization of exponential function.

Definition 1.2. A two parameter function of Mittag-Leffler type is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0.$$

During the first half of the twentieth century, the Mittag-Leffler functions remained almost unknown to the majority of the mathematicians. However, recently attention of mathematicians and other scientists towards functions of Mittag-Leffler type has increased and today some mathematicians like to refer the classical Mittag-Leffler function as the *Queen Function* of Fractional Calculus, and to consider all related functions as her court.

Recently, a complete monograph (see Ref. [8]) about Mittag-Leffler functions and its several applications has been published. More information about this kind of special functions ans its relations with Fractional Calculus can be found in Ref. [9].

It follows from previous definitions that, for instance,

$$E_{1,1}(z) = \exp(z), \qquad E_{2,1}(z^2) = \cosh(z),$$
$$E_{1,2}(z) = \frac{\exp(z) - 1}{z}, \qquad E_{2,2}(z^2) = \frac{\sinh(z)}{z},$$
$$E_{1,3}(z) = \frac{\exp(z) - 1 - z}{z^2}, \qquad E_{2,2}(z^2) = \cos(z),$$

which proves the importance of Mittag-Leffler functions in mathematics. Furthermore, using the following well known result

$$\left[{}^{\mathrm{c}}D^\alpha x^b\right](t) = t^{b-\alpha} \frac{\Gamma(b+1)}{\Gamma(b+1-\alpha)}, \quad b > -1,\ b \neq 0,$$

we can easily see that $E_\alpha(\lambda t^\alpha)$, with $\lambda \in \mathbb{R}$, is a nontrivial eigenfunction for Caputo fractional derivative operator (previously denoted by ${}^{\mathrm{c}}D^\alpha$, see Ref. [5, Section 2.4]). That is, if $f(t) = E_\alpha(\lambda t^\alpha)$ for $t \geq 0$ then

$${}^{\mathrm{c}}D^\alpha f(t) = \lambda f(t), \quad t \geq 0.$$

In the past, some authors used to derived some results the following *property* of Mittag-Leffler function

$$E_\alpha(a(t+s)^\alpha) = E_\alpha(at^\alpha) E_\alpha(as^\alpha),$$

where a is a real constant. Unfortunately, such property is unavailable unless $\alpha = 1$ or $a = 0$, both of them are trivial situations, since we obtain exponential function in the first case, and a constant function secondly. For more information about the invalidity of such equality we refer to Ref. [10].

In this note we discuss a similar property related again with Mittag-Leffler function and using such property, we conclude that the function introduced in Ref. [11] is not an exact solution for the fractional logistic equation. It may be obvious, but nevertheless Mittag-Leffler function shares some analogue properties with exponential function, these kind of results show that exponential function cannot be replaced, in general, by Mittag-Leffler function in the theory of Fractional Calculus.

2. Main results

In 1838, P.F. Verhulst introduced a nonlinear term into the rate equation; he was studying population models and he wanted to avoid the catastrophic predictions previously proposed by T. Malthus, who had used the rate equation to model human population growth (for historical references see Ref. [11]). By this way, P.F. Verhulst obtained what today is known as the logistic equation:

$$u'(t) = ku(t)(1 - u(t)), \quad t \geq 0. \tag{1}$$