Physica A 444 (2016) 905–913

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Entropic measures of joint uncertainty: Effects of lack of majorization

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HIGHLIGHTS

- Different Renyi entropies lead to contradicting uncertainty relations.
- Contradicting uncertainties are explained as lack of majorization of statistics.
- The comparison between joint and product distributions depends on purity.
- Most popular measures of complementarity are blind this these features.

ARTICLE INFO

Article history:
Received 17 June 2015
Received in revised form 11 September 2015
Available online 27 October 2015

Keywords:
Quantum uncertainty
Rényi entropies
Majorization
Complementarity

ABSTRACT

We compute Rényi entropies for the statistics of a noisy simultaneous observation of two complementary observables in two-dimensional quantum systems. The relative amount of uncertainty between two states depends on the uncertainty measure used. These results are not reproduced by a more standard duality relation. We show that these behaviors are consistent with the lack of majorization relation between the corresponding statistics.

1. Introduction

Historically, the joint uncertainty of pairs of observables has been mostly addressed in terms of the product of their variances. Nevertheless, there are situations where such formulation is not satisfactory enough [1], thus alternative approaches have been proposed, mainly in terms of diverse entropic measures [2–5] (see also the reviews in Ref. [6]). In this work we consider in particular the so-called Rényi entropies [7] and the corresponding entropic uncertainty relations, for the statistics associated to two complementary observables [8]. There has been an increasing activity to obtain different and improved entropic uncertainty relations not only for foundational reasons but also for the different applications in quantum information problems (a non-exhaustive list includes information-theoretic formulation of error–disturbance relations [9], connection with duality relations [10] and nonlocality [11], entanglement detection [12], EPR-steering inequalities [13], quantum memory [14], and security of quantum cryptography protocols [15]). Also, entropic uncertainty relations have a deep connection

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http://dx.doi.org/10.1016/j.physa.2015.10.097
0378-4371/

with the majorization of statistical distributions [16,17], which has been already applied to examine uncertainty of thermal states [18] (this is closely related to the idea of mixing character [19]).

However, previous works [4,8] have shown that entropic uncertainty relations may lead to unexpected results, derived from the fact that the amount of uncertainty for a pair of observables depends on the uncertainty measure used. This is quite natural; actually, one of the benefits of using entropic measures is that they adapt to assess different operational tasks. Nevertheless, one may find it surprising that different measures lead to opposite conclusions in entropic relations: this is, that the states of maximum uncertainty for one measure are the minimum uncertainty states for the other measure, and vice versa.

In this regard, the aim of this work is twofold. On the one hand, we show that these unexpected behaviors are fully compatible with the lack of majorization relation between the corresponding statistics. This connection holds because entropic measures are monotone with respect to majorization. Thus, such surprising entropic results are not tricky features of entropic measures, but may have a deeper meaning that is actually overlooked by more popular measures of uncertainty or complementarity. On the other hand, we extend the application of entropic measures to the statistics of a simultaneous joint observation of two complementary observables in the same system realization [20–22]. This setting of complementarity in practice provides a rich arena to examine the interplay between entropic measures and majorization. The simultaneous measurement provides a true joint classical-like probability distribution that enables alternative assessments of joint uncertainty, different from the ones given by the product of individual statistics, either intrinsic or of operational origin.

For simplicity we address these issues in the simplest quantum system described by a state in a two-dimensional Hilbert space. This comprises very relevant practical situations such as the path–interference complementarity in two-beam interference experiments. This allows us to contrast the performance of entropic measures with respect to more standard descriptions of complementarity [23–25].

The paper is organized as follows: in Section 2 we introduce the discussion on statistics of simultaneous measurements for spin 1/2 observables. Section 3 exhibits noticeable results for entropic quantities, and an explanation for that behavior is given in Section 4. In Section 5, a duality relation for complementarity is analyzed and compared with the entropic results. Finally, some concluding remarks are outlined in Section 6.

2. Statistics and simultaneous measurements

Let us consider two complementary observables represented by the Pauli spin matrices σ_x and σ_z. In practical terms they may represent phase and path, respectively, in two-beam interference experiments. The system state is described by a density matrix operator acting on the Hilbert space $\mathcal{H}_S$ that in Bloch representation acquires the form $\rho = \frac{1}{2}(I + \boldsymbol{s}\cdot\boldsymbol{\sigma})$, where I is the identity matrix, $\boldsymbol{\sigma}$ represents the three Pauli matrices, and $\boldsymbol{s} = \mathrm{Tr}(\rho\,\boldsymbol{\sigma})$ is a three-dimensional Bloch vector with $|\boldsymbol{s}| \le 1$. The modulus $|\boldsymbol{s}|$ expresses the degree of purity of the state as $\mathrm{Tr}(\rho^2) = \frac{1}{2}(1 + |\boldsymbol{s}|^2)$, being $|\boldsymbol{s}| = 1$ in the case of a pure state. We make use of the Bloch-sphere parametrization:

$$s_x = |\boldsymbol{s}| \sin\theta\cos\varphi, \qquad s_y = |\boldsymbol{s}| \sin\theta\sin\varphi, \qquad s_z = |\boldsymbol{s}|\cos\theta. \tag{1}$$

The *intrinsic statistics* for the observables σ_x and σ_z are

$$p_j^X = \frac{1}{2}(1 + j\,s_x) \quad \text{and} \quad p_k^Z = \frac{1}{2}(1 + k\,s_z), \tag{2}$$

with $j = \pm 1$ and $k = \pm 1$.

The *simultaneous measurement* of noncommuting observables requires involving auxiliary degrees of freedom, usually referred to as apparatus. In our case we consider an apparatus described by a two-dimensional Hilbert space $\mathcal{H}_A$. The measurement performed in $\mathcal{H}_A$ addresses that of σ_z, while σ_x is measured directly on the system space $\mathcal{H}_S$. The system–apparatus coupling transferring information about σ_z from the system to the apparatus is arranged via the following unitary transformation acting on $\mathcal{H}_S \otimes \mathcal{H}_A$,

$$U = |+\rangle\langle+| \otimes U_+ + |-\rangle\langle-| \otimes U_-, \tag{3}$$

where $U_\pm$ are unitary operators acting solely on $\mathcal{H}_A$, while $|\pm\rangle$ are the eigenstates of σ_z with corresponding eigenvalues ± 1. For simplicity the initial state of the apparatus, $|a\rangle \in \mathcal{H}_A$, is assumed to be pure, so that the system–apparatus coupling leads to

$$U|+\rangle|a\rangle \to |+\rangle|a_+\rangle, \qquad U|-\rangle|a\rangle \to |-\rangle|a_-\rangle, \tag{4}$$

where the states $|a_\pm\rangle = U_\pm|a\rangle \in \mathcal{H}_A$ are not orthogonal in general, with $\cos\delta = \langle a_+|a_-\rangle$ assumed to be a positive real number with $0 \le \delta \le \pi/2$, without loss of generality. The measurement in $\mathcal{H}_A$ introducing minimum additional noise is given by projection on the orthogonal vectors $|b_\pm\rangle$ (see Fig. 1):

$$\begin{aligned} |b_+\rangle &= \frac{1}{\cos\phi}\left(\cos\frac{\phi}{2}|a_+\rangle - \sin\frac{\phi}{2}|a_-\rangle\right), \\ |b_-\rangle &= \frac{1}{\cos\phi}\left(-\sin\frac{\phi}{2}|a_+\rangle + \cos\frac{\phi}{2}|a_-\rangle\right), \end{aligned} \tag{5}$$