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"Phase diagram" of a mean field game

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HIGHLIGHTS

- We study a simple model of "mean field game".
- We provide an exact solution of the associated system of coupled differential equations.
- We analyze the resulting self-consistent equation in various limiting regimes, resulting in the construction of a "phase diagram" of the considered mean field game.

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ABSTRACT

Mean field games were introduced by J-M. Lasry and P-L. Lions in the mathematical community, and independently by M. Huang and co-workers in the engineering community, to deal with optimization problems when the number of agents becomes very large. In this article we study in detail a particular example called the "seminar problem" introduced by O. Guéant, J-M. Lasry, and P-L. Lions in 2010. This model contains the main ingredients of any mean field game but has the particular feature that all agents are coupled only through a simple random event (the seminar starting time) that they all contribute to form. In the mean field limit, this event becomes deterministic and its value can be fixed through a self consistent procedure. This allows for a rather thorough understanding of the solutions of the problem, through both exact results and a detailed analysis of various limiting regimes. For a sensible class of initial configurations, distinct behaviors can be associated to different domains in the parameter space. For this reason, the "seminar problem" appears to be an interesting toy model on which both intuition and technical approaches can be tested as a preliminary study toward more complex mean field game models.

1. Introduction

Many problems in different fields deal with a situation where many identical and interacting agents try to minimize a cost through the choice of a strategy. One can think of economic agents trying to maximize their profits, of people in a crowd trying to minimize their discomfort or to particles in a fluid "trying" to minimize their energy.

A general framework making possible to model a large class of such problems has been introduced in 2006 by Lasry and Lions [1,2] and Huang et al. [3] under the general terminology of "mean field game theory". Largely inspired by statistical physics, this approach addresses the limit where the agents face a continuum of choices (states) in which they can evolve only locally, and the number of agents is large enough that self averaging processes are at work. This approach leads to a system of partial differential equations coupling the density of players and the optimization part of the problem.

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Mean field game theory has been intensively studied in the past few years, and in spite of its relative youth, a very large number of results have been obtained in the mathematical [4–8] and socio-economic communities [9–13]. A recent overview is given by Gomes and Saúde in Ref. [14]. Most of the focus however has been put either on the conditions required to prove rigorously the existence and unicity of the solutions of the equations of mean field game theory [8], or on the study of particular models based primarily on numerical treatments [7]. A more "qualitative" understanding of the behavior of the solutions, based on the identification of the relevant time and length scales, and on the analytical study of the solution in various limiting regime, has received significantly less attention.

Our goal in this paper is to perform this program for a simple model, introduced by Guéant et al. in 2010 [15], called the seminar problem to be described in more details below. The essential point here is that this "mean field game model" is in some sense very close to the everyday "physicists' mean field" since all agents are interacting only through a very simple "field" which is actually a simple number, the time T at which the seminar actually starts. This particular feature allows for an analytical approach, similar in spirit to the physicists' one: For fixed T, the behavior of each agent becomes independent on the other, making the associated problem to be solvable to a large extent; then, for a given distribution of agents, the actual value of T can be evaluated by a self-consistency procedure. The main interest in this model is to provide a fully understandable toy model on which one can develop its own intuition and tools before tackling the full complexity of mean field game models.

The paper is organized as follows: In Section 2 we introduce the seminar problem in detail and show that its resolution involves two essentially independent parts: a system of coupled (Hamilton–Jacobi–Bellman and Kolmogorov) differential equations on one hand, and a self-consistency problem on the other. Sections 3 and 4 address the Hamilton–Jacobi–Bellman and Kolmogorov equations, respectively. Various limiting regimes are studied in detail for both. Moreover, we show that an exact solution to these coupled differential equations can actually be given in a closed form. The self-consistency condition determining the effective beginning of the seminar T is discussed in Section 5, eventually leading to the construction of a "phase diagram" for this toy model. Concluding remarks are gathered in Section 6. The paper is completed by three Appendices where technical computations are shown.

2. The seminar problem

The model

Consider a corridor at the end of which is a seminar room. A seminar is planned at time $\bar{t}$ but people know that in practice, it will only begin when a large enough proportion of the lab members θ (known), will be seated.

The members of the laboratory thus move according to the following considerations: They do not want to arrive too early in the seminar room because they do not particularly enjoy waiting idly as the room fills. On the other hand they are aware that the lab director and the seminar organizers will already be in the room at time $\bar{t}$, and will frown upon late comers. Furthermore they really want to understand the content of the seminar and are concerned that missing the actual beginning might not help in this respect.

For every agent, this is summarized by the following cost function associated with the arrival time t:

$$c(t) = \alpha[t - \bar{t}]_+ + \beta[t - T]_+ + \gamma[T - t]_+, \tag{1}$$

where T is the effective beginning time of the seminar. In Eq. (1), α, β and γ are positive real numbers and respectively quantify the sensitivity to social pressure, the desire not to miss the beginning of the seminar, and the reluctance to useless waiting. We assume these parameters to be the same for all members of the laboratory. We also assume ($\gamma < \alpha$) so that the cost $c(t)$ is actually minimal for the official starting time $\bar{t}$.

The corridor is represented by the negative half-line line $]-\infty, 0]$, and the seminar room is located at $x = 0$. At time $t = 0$, people leave their office to go to the seminar. Each member of the laboratory $i = 1 \ldots N$, controls her drift $a_i(t)$ toward the seminar room but is subject to random perturbations (stopping to discuss with somebody, going back to take a pen and then giving up the idea, or speeding up to catch up a friend for example), modeled by a Gaussian white noise of variance σ^2. A given participant thus moves according to a noisy dynamics:

$$\mathrm{d}X_i = a_i(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_i(t) \tag{2}$$

where,

$X_i(t)$ is the agent position at time t,

$a_i(t)$ is her drift at the same time,

$\mathrm{d}W_i(t)$ is a normal white noise.

Again, except for their initial positions, all agents have the same characteristics.

In addition to the cost $c(t)$ associated to the arrival time (Eq. (1)), agents dislike having to rush on their way to the seminar room and the total cost function therefore includes a term quadratic in the (controlled) drift $a_i(t)$. An agent leaving her workplace x_0 at $t = 0$ has thus to adapt her drift in order to minimize the expected cost

$$J_T[a] = \mathbb{E}\left[c(\tilde{\tau}) + \frac{1}{2}\int_0^{\tilde{\tau}} a_i^2(\tau)\,\mathrm{d}\tau\right] \tag{3}$$