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Analysis of ground state in random bipartite matching

Gui-Yuan Shi [a], Yi-Xiu Kong [a], Hao Liao [b,*], Yi-Cheng Zhang [a]

[a] *Physics Department, University of Fribourg, Chemin du Musée 3, CH-1700 Fribourg, Switzerland*

[b] *Guangdong Province Key Laboratory of Popular High Performance Computers, College of Computer Science and Software Engineering, Shenzhen University, Nanhai Road 3688, 518060, Shenzhen, China*

HIGHLIGHTS

- We applied the Kuhn–Munkres Algorithm to analyze ground state of bipartite matching problem.
- We settle down the quantity and distribution of blocking pairs in the ground state.
- The stability of ground state decreases exponentially with better connectivity in the network.
- The scope of application of the ground state is extended to a broader initial conditions.

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ABSTRACT

Bipartite matching problems emerge in many human social phenomena. In this paper, we study the ground state of the Gale–Shapley model, which is the most popular bipartite matching model. We apply the Kuhn–Munkres algorithm to compute the numerical ground state of the model. For the first time, we obtain the number of blocking pairs which is a measure of the system instability. We also show that the number of blocking pairs formed by each person follows a geometric distribution. Furthermore, we study how the connectivity in the bipartite matching problems influences the instability of the ground state.

1. Introduction

Bipartite matching problems, appear in many social processes like the marriage problem between men and women, college admission problem between students and universities, assignment between workers and jobs, and also the choice making between buyers and sellers. Due to its various applications in the real world, not only economists but also the statistic physicists are attracted by the bipartite matching problems.

Gale and Shapley first introduced the stable marriage problem, a one-to-one two side matching [1], which is the most important bipartite matching problem. Bipartite matching problems was rephrased to an optimization problem by assigning agent i an energy term ε_i to represent his/her satisfaction. The so-called optimal matching has the minimum energy among all the possible matchings under certain assumptions. And a matching is called stable only if there are no two agents man i and woman α, each of whom prefer the other to their spouse in x. Such a pair is said to be a blocking pair [2] with respect to x, abbreviated as *BP* hereafter. Gale–Shapley algorithm proves the existence of a stable solution in the matching problem under any circumstance. By neglecting the stability of the state, Mézard and Parisi applied the replica method of spin glass theory to study the global optimal solution of one-to-one two side matching problem [3,4]. Afterward, Zhang et al. studied the scaling behavior and partial information matching of the marriage problem [5–7]. Furthermore, Dzierzawa and Oméro introduced

* Corresponding author.
E-mail addresses: jamesliao520@gmail.com (H. Liao), yi-cheng.zhang@unifr.ch (Y.-C. Zhang).

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Table 1
Using of the variables in this paper.

Variable	Description
n	The number of male agents or female agents
$\mathscr{M}$	The set of male agents
$\mathscr{W}$	The set of female agents
x	A matching $x : \mathscr{M} \to \mathscr{W}$
$M_{i,\alpha}$	The energy of man m_i if woman w_α was matched to him
$W_{\beta,j}$	The energy of woman w_β if man m_j was matched to her
$H_{i,\alpha}$	The mean energy of man m_i and woman w_α if man m_i and woman w_α were matched
$E_H(x)$	Average energy per person under a given matching x
$E_H(GS)$	Average energy per person for the ground state
$\langle E_H(GS)\rangle$	Mean value of $E_H(GS)$
$f(\epsilon)$	Probability distribution function of individual energy in the ground state
P_k	The probability that a man forms k BPs
N_{BP}	The number of BPs
N_{bp}	The number of men/women who form BPs (note that a person may have more than one BP)
σ	The fraction that the number of persons one can know in all the agents of the other sex
S_σ	The probability of a person could not find a BP under the constraint of a person only knows $n\sigma$ persons of the other sex.

the acceptance threshold and thus improved the matching result [8]. Recently, Zhou et al. studied the bidirectional selection problems from a new perspective of social networks [9,10]. In this paper, we aim to compute the number of blocking pairs in the global optimal solution and then analyze the properties of this solution.

The rest of this paper is organized as follows. In Section 2, we introduce the Gale–Shapley model. In Section 3, we use Kuhn–Munkres algorithm to find the global optimal solution of the bipartite matching problem. In Section 4, we analyze the blocking pairs in the ground state, with both numerical and analytical approaches. Then, we analyze the stability of the ground state based on the number of blocking pairs.

2. Model

The Gale–Shapley model is a matching model in which there are two sets of participants, men and women. We denote by $\mathscr{M} = \{m_1, m_2, \ldots, m_n\}$ and $\mathscr{W} = \{w_1, w_2, \ldots, w_n\}$, the set of men and women, respectively. A matching is a one-to-one mapping between the two disjoint sets, *i.e.*, an invertible bijection $x : \mathscr{M} \to \mathscr{W}$. A matching x can be denoted as:

$$x = [(m_1, x(m_1)), (m_2, x(m_2)), \ldots, (m_n, x(m_n))],$$

where $x(m_i) = w_\alpha$ means the woman who matched with man m_i, and $x^{-1}(w_\alpha) = m_i$ means the man who matched with w_α [11].

Previous works [3,7,8] assumed a discrete uniform distribution for the energy term $\varepsilon = 1, 2, \ldots, n$, but we hold that the energy should be independent of the model size n. Therefore we suggest that the energy ε_i follows a continuous uniform distribution on [0, 1], and result will be consistent with previous work [3,7] since the size of the model is large enough to eliminate the difference between discrete and continuous distribution.

For each matched pair (m_i, w_α), we denote by $M_{i,\alpha}$ and $W_{\alpha,i}$ the energy of man i and woman α respectively. We denote by M and W, the matrices composed of the energies $M_{i,\alpha}$ and $W_{\alpha,i}$, respectively. For a given matching $x = [(m_1, x(m_1)), (m_2, x(m_2)), \ldots, (m_n, x(m_n))]$, the average energy per person $E_H(x)$ is

$$E_H(x) = \frac{1}{2n}\left[\sum_{i=1}^{n} M_{i,x(m_i)} + \sum_{i=1}^{n} W_{x(m_i),i}\right] = \frac{1}{n}\sum_{i=1}^{n} H_{i,x(m_i)}, \tag{1}$$

where we defined the mean energy of man i and woman α $H_{i,\alpha} = \frac{1}{2}(M_{i,\alpha} + W_{\alpha,i})$. We denote by $E_H(GS)$ the energy of the ground state x_{GS} of the model, which is the matching x_{GS} that minimizes the energy $E_H(x)$.

Table 1 summarizes the notations adopted in this paper for the variables of the model.

3. Energy analysis

We apply the Kuhn–Munkres algorithm [12,13] to find the ground state of the model. The Kuhn–Munkres algorithm is described in Table 2. We study the properties of the ground state when changing the size n of the model.

Fig. 1 shows how the average energy $\langle E_H(GS)\rangle$ of 100 realizations of ground state depends on the size n of the model. Numerical simulation results were fitted with a power-law $\langle E_H(GS)\rangle = An^{-\beta}$ with the least square method. The estimated exponent is $\beta = 0.50$, the expected average energy per person is $\langle E_H(GS)\rangle = \frac{0.808}{\sqrt{n}}$, which is consistent with the result in Ref. [7] as we expected.