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# A topological theorem and correlations, within the context of stochastic evolution

E.F. Costanza [a], G. Costanza [b,*]

[a] Departamento de Matemática, Facultad de Matemática, Astronomía y Física (FAMAF), Universidad Nacional de Córdoba, Av. Medina Allende s/n, Ciudad Universitaria, CP:X5000HUA Córdoba, Argentina

[b] Departamento de Física, Instituto de Física Aplicada (INFAP), Universidad Nacional de San Luis, Chacabuco 917, 5700 San Luis, Argentina

## HIGHLIGHTS

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## ABSTRACT

A topological theorem, that proves that any d-dimensional lattice is equivalent to a one-dimensional one, allows to write the evolution equations as a function of only one spatial coordinate. Stochastic and continuum deterministic evolution equations, are derived from a set of discrete stochastic evolution equations. The evolution equations of the dynamical variables and correlations are obtained for processes that evolve non-Markovianly. Some relatively simple examples are given in order to illustrate the procedures.


## 1. Introduction

During many decades a lot of work was dedicated to develop different approaches and technics that allow to find the evolution equations of the dynamical variables that describe the problem at hand. Illustrative examples, some of them pioneers in this field, can be found in Refs. [1–21]. Recently, an approach that allows to derive deterministic evolution equations from a set of stochastic evolution equations, after an average over realizations, with both multiplicative as well as additive noises, was derived in Refs. [22–27]. In those papers different aspects were developed in order to show the versatility of the approach. In the present approach two results will be obtained allowing to handle, in a simplified way, some problems that arise when the dimensionality of the system is greater than one. To transform problems of dimensionality greater than one to a one-dimensional problem a topological theorem will be proved that allows to reduce a d-dimensional problem to a one-dimensional one. This circumstance usually simplifies matters considerably. Equipped with this result, formulas for correlations of dynamical variables that are complex numbers will be developed for updating rules that define the model capable of describing Markovian as well as Non-Markovian evolution, with noises that can be multiplicative or additive or both. Some quantum mechanical simple examples, including a spinor-like set of two linear couple evolution

* Corresponding author.
*E-mail address:* costanza@unsl.edu.ar (G. Costanza).

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equations, are studied and the evolution equations for the correlations are obtained. Also, a model similar to the one obtained by Parmeggiani et al. [28], will be considered in order to show an example of a system where the evolution equations of the dynamical variables are non-linear. This model shows an example of nonlinear evolution equations for the dynamical variable coupled to the correlation. These relatively simple illustrative examples show the basic procedure necessary for find the evolution equations for the dynamical variable as well as the correlations.

The paper is organized as follows. In Section 2 a topological theorem will be proved that demonstrates that a d-dimensional lattice is topologically equivalent to a one-dimensional lattice allowing to simplify the remaining of the paper in the sense that it is possible to consider one-dimensional problems without loss of generality. In Section 3 an introduction of the stochastic evolution rules corresponding to models with an updating of the dynamical variables that depend on the values of the dynamical variables at an arbitrary number of previous time steps and with subsets that belong to different types, is considered. General formulas for the evolution equation as well as correlations will be obtained. In Section 3 some simple illustrative examples are developed with certain degree of detail in order to provide skills in working with this approach. Finally, in Section 4, the general features and conclusions together with possible extensions are given.

## 2. A topological theorem

An interesting topological result will be summarized in a theorem that states that all as large as we please but finite lattices are topologically equivalent to a one-dimensional lattice. Before the statement of the theorem some remarks on the jargon or parlance usually used in graph theory are necessary. A lattice can be visualized as a graph with vertices or points connected by edges or segments between them. An example can be seen in Fig. 1(a). This example shows a graph with five vertices or points and seven edges or segments that connect pairs of points or, as one of the edges on vertex five, connecting a point with itself. With the above introduced language it is possible to state and prove the following theorem.

**Theorem.** *A d-dimensional lattice represented by a graph consisting of vertices and edges is topologically equivalent to a one-dimensional lattice.*

**Proof.** The proof is obtained in two steps in a very simple way. First, numbering each point of the graph representing a d-dimensional lattice (in the example a two-dimensional lattice) as in Fig. 1(a). Second, after representing each of the points in an axis and the edges as segments connecting the points of the new lattice, as shown in Fig. 1(b), the new lattice has the same elements (vertices) and connections (edges) as the original one. With this procedure was constructed a one-dimensional lattice that is topologically equivalent to the original lattice. This procedure can be done on an arbitrary lattice and the theorem is proved in a very simple and intuitive way. □

It is interesting to note that the numbering used could be changed to other different set of numbers producing also a one-dimensional lattice. As some examples, could be used the following set of points: $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$, producing a one-dimensional lattice with the first point at the origin and a symmetrical representation with respect to the origin, respectively. Another remark, that is in order here, is that the theorem proved above will be used below to construct evolution equations of a set of dynamical variables that are sets of interpolating functions $\{q(t, x)\}$ that take the discrete values $\{q_{i_0,i_1}\}$. An example with one interpolating function $q(t, x)$, shown in Fig. 3, that takes the discrete values $q_{i_0,i_1}$, shown in Fig. 2, is described in a Cartesian coordinates representation in order to illustrate the smoothing procedure.

## 3. The non-Markovian discrete stochastic evolution updating: basic definitions

In Ref. [27] the basic definitions for a non-Markovian discrete stochastic evolution updating were given and reproduced below, with the appropriate modifications, for the sake of completeness and in order to provide the basic definitions used in the next sections. In order to make the presentation as simple as possible, and due to the results of the theorem proved above, a one-dimensional lattice $\Lambda$ with periodic boundary conditions in an interval $[-L_0/2, +L_0/2]$ ($L_0$ being as large as we please but finite) will be considered and a set of complex dynamical variables $\{q_s^{(r)}(t, x)\}$ will be used for describing the value of each dynamical variable in a realization $r$, in a state or type of dynamical variable $s$, at spatial coordinate $x = x_{i_1}$ and at time $t = t_{i_0}$. Note that here, additional indices like $r$ and $s$ were used, that were dropped in the previous sections and in the figures, in order to save printing and added here due to the necessity of describing evolution equations that are stochastic and of different types. The spatial coordinate and time correspond to the generic site $i_1$ and to the update $i_0$, respectively. $s$ designates the generic value of the set $\{1, \dots, S\}$, where $S$ is the number of elements of the set. The spacing between sites or lattice constant is $a_1$ and the time between two successive updates is $a_0$. In order to save printing and without loss of generality, both or one of the two constants will be set equal to one when it is suitable. The number of lattice sites is $M = L_0 + 1$ and the length of the lattice is $L = a_1(M - 1)$. The evolution equation for the set of dynamical variable can be expressed, as in Ref. [27], in the following general form

$$q_s^{(r)}(t + a_0, x) = q_s^{(r)}(t, x) + G_s^{(r)}(t, \dots, t - l_k a_0, X_{l_{01}}, \dots, X_{l_{0k}}, X_j, X_\xi), \quad \forall s \in \{1, \dots, S\},\ t \geq 0,\ x \in \Lambda, \tag{1}$$

where $G$ denotes the set of updating rules that define a given model and $X_{l_{01}}, \dots, X_{l_{0k}}$ denote the set of complex dynamical variables $\{q_s^{(r)}(t, x)\}, \dots, \{q_s^{(r)}(t - l_{0k} a_0, x)\}$, respectively. The sets of both discrete and continuous stochastic variables that